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Khovanov homology of links and graphs

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Resumo

Nesta tese, trabalhamos com a homologia de Khovanov de enlaces e com a homologia de grafos. A homologia de Khovanov de enlaces consta dum complexo graduado de cadeias que é um invariante de enlace, cuja característica de Euler é igual ao polinómio de Jones do enlace, e portanto pode ser vista como a "categorificação" do polinómio de Jones.

Provamos que o primeiro grupo de homologia dum nó de trança positiva (*positive braid knot*) é trivial. Depois, provamos que os nós de toro, não alternados são homologicamente espessos (*thick*). Também, demonstramos que podemos diminuir o número de torções completas de nós de toro sem mudar a homologia de baixo grau, implicando a existência de homologia estável de nós de toro. Mais, provamos a maior parte das propriedades anteriores também para a homologia de Khovanov-Rozansky.

Na homologia de grafos, categorificamos o polinómio dicromático (e em consequência, o polinómio de Tutte) para grafos, através da categorificação dum conjunto infinito de especializações duma variável. Também, categorificamos explicitamente a especialização que é analoga ao polinómio de Jones dum enlace alternado correspondente ao grafo inicial. Mais, categorificamos explicitamente o polinómio dicromático completo de duas variáveis de grafos, usando os complexos de Koszul.

Palavras-chave: homologia de Khovanov, polinómio de Jones, enlace, nós de toro, grafo, polinómio dicromático

Abstract

In this thesis we work with Khovanov homology of links and its generalizations, as well as with the homology of graphs. Khovanov homology of links consists of graded chain complexes which are link invariants, up to chain homotopy, with graded Euler characteristic equal to the Jones polynomial of the link. Hence, it can be regarded as the "categorification" of the Jones polynomial.

We prove that the first homology group of positive braid knots is trivial. Futhermore, we prove that non-alternating torus knots are homologically thick. In addition, we show that we can decrease the number of full twists of torus knots without changing low-degree homology and consequently that there exists stable homology for torus knots. We also prove most of the above properties for Khovanov-Rozansky homology.

Concerning graph homology, we categorify the dichromatic (and consequently Tutte) polynomial for graphs, by categorifying an infinite set of its one-variable specializations. We categorify explicitly the one-variable specialization that is an analog of the Jones polynomial of an alternating link corresponding to the initial graph. Also, we categorify explicitly the whole two-variable dichromatic polynomial of graphs by using Koszul complexes.

Key-words: Khovanov homology, Jones polynomial, link, torus knot, graph, dichromatic polynomial

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Contents

Chapter 1

Introduction

One of the most interesting and promising recent developments in mathematics in general is the latest construction in knot theory called the "categorification" of link invariants, initiated by M. Khovanov in [24]. This construction opened new horizons by providing tools for approaching and solving various problems in knot theory and low-dimensional topology. Since the topic is very new, the frontiers of its applications are still far away, and its potential is enormous.

1.1 Brief historical review of polynomial link invariants

Mathematical properties of knots – circles smoothly embedded in 3-space – have been studied for a long time. Although the knots are 3-dimensional objects, there is a nice way to present them in two dimensions by a planar projection (also called a regular diagram). The first attempts to tabulate knots date from the late XIX century. The knots and links (finite disjoint collection of knots) are naturally identified by isotopy. Hence in order to distinguish two knots (or links) one needs an invariant, i.e. a function on the set of knots which has the same value on isotopic knots.

A first step toward finding knot invariants was made by K. Reidemeister in [45], who proved that two planar projections represent isotopic knots (or links) if and only if one can be obtained from the another by the finite application of the famous Reidemeister moves. Hence, a function on the set of planar diagrams will be a link invariant if and only if it is invariant under Reidemeister moves.

Even before the discovery of the Reidemeister moves, in 1923 in [1],

Alexander defined geometrically (by working in 3-space) a polynomial link invariant which was very good at distinguishing knots. Later, in 1970 in [9], Conway showed that the Alexander polynomial can also be defined combinatorially by working with planar projections and that it satisfies a certain skein relation.

In 1984, V.F.R. Jones revolutionarized knot theory by defining a polynomial invariant which satisfies a different skein relation to the Alexander-Conway one. The original Jones construction uses von Neumann algebras and is rather complicated ([20]). However, in 1987, L. Kauffman in [23] introduced a state-sum model construction of the Jones polynomial which is purely combinatorial and remarkably simple. Although the defining relations of the Alexander-Conway and Jones polynomials are very similar, their properties are quite different. For instance, the Jones polynomial can distinguish knots from their mirror images, and it was used to prove some old Tait conjectures on regular diagrams of alternating links ([39], [22]).

Soon after the discovery of the Jones polynomial there was an explosion of generalizations, which culminated in the two-variable HOMFLYPT polynomial named after its many dicoverers ([12], [41]). It contains the Alexander-Conway and Jones polynomial as specializations and it is the most general polynomial invariant that possesses a skein relation. Alternatively, the two-variable HOMFLYPT polynomial can be given by an infinite sequence of one-variable specializations, one for each positive integer n, which is enough to recover the whole HOMFLYPT polynomial. Finally, in [38] a state-sum model was obtained for the n-specializations of the HOMFLYPT polynomial.

As was shown in the late 80's, [46], [54], the n-specializations of the HOMFLYPT polynomial can be obtained by using the representation theory of the quantum $sl(n)$ group. Hence, the n-specializations of the HOMFLYPT polynomial are also called the $sl(n)$ link polynomials. Also, the homology theories that categorify them (see the following section), i.e. Khovanov homology for the $sl(2)$ case (see Section 2.4), and Khovanov-Rozansky homology for the general $sl(n)$ case (see Section 2.6), are also called $sl(2)$-link homology and $sl(n)$-link homology, respectively.

1.2 Link homology

In 1999, M. Khovanov in [24] gave a completely new perspective on link polynomials. For each link L in S^3 he defined a graded chain complex, with grading preserving differentials, whose graded Euler characteristic is

equal to the Jones polynomial of the link L. This is done by starting from the state-sum expression for the Jones polynomial (which is written as an alternating sum), then constructing for each term a graded module whose graded dimension is equal to the value of that term, and finally, defining the differentials as appropriate grading preserving maps, so that the complex obtained is a link invariant (up to chain homotopy). It has also been shown by M. Khovanov in [26] and in a different way by M. Jacobsson in [18], that the homology groups are functorial, i.e. that there exists a functor from the category of links as objects and cobordisms between them (surfaces in 4-space), to the category of abelian groups, such that on links it is given by the homology of the Khovanov complex. Also, Khovanov homology is strictly stronger than the Jones polynomial, i.e. there exist knots with the same Jones polynomial and different Khovanov homology ([5]). The situation is similar to the Euler characteristic of a manifold: it can be seen as an alternating sum of the dimensions of the homology groups of a manifold, which are functorial, and which are stronger invariants than the Euler characteristic itself.

Although the theory is rather new, it already has strong applications in low-dimensional topology. For instance, the short proof of the Milnor conjecture by Rasmussen in [42], as well as the proof of the existence of exotic differential structures on $\mathbb{R}^4$ ([43], [13]), which were previously accessible only by gauge theory ([32], [10]).

The advantage of Khovanov homology theory is that its definition is combinatorial and since there is a straightforward algorithm for computing it, it is (theoretically) highly calculable. Nowadays there are several computer programs [7], [52] that can calculate effectively Khovanov homology of links with up to 50 crossings.

Based on the calculations there are many conjectures about the properties of link homology, see e.g. [5], [28], [11]. Some of the properties have been proved by now (see [33], [34]), but many of them are still open.

In this thesis we solve some conjectures concerning Khovanov homology. First, in Chapter 5, we prove the following conjecture from [28]:

Theorem 18: *The first homology group of a positive braid knot is trivial.*

The Khovanov homology of a link is naturally bigraded, and it is common practice to represent it in a planar array, by writing at the position (i, j) the rank of the homology group $\mathcal{H}^{i,j}(L)$. As shown by calculations, the nonzero entries are grouped along the diagonals. The homology of every

knot occupies at least two diagonals, and the knots whose homology occupy exactly two diagonals are called homologically thin. All other knots are called homologically thick.

The torus knots – the knots that can be drawn on a surface of a solid torus without intersection – constitute an important class of links. They are classified up to isotopy by two integers, p and q. Every torus knot is a positive braid knot. Hence, their first homology group is trivial by Theorem 18. In Chapter 6 we also obtain the following:

Corollary 21: *The torus knots $T_{p,q}$, for $3 \leq p \leq q$ are homologically thick.*

Moreover, in the course of the proof, we obtain relations between the homology groups of the torus knot $T_{p,q}$ and the torus knot $T_{p,q-1}$ (see Theorem 23 in Chapter 6). As a consequence, this gives low-degree homology groups of torus knots and also a stability property for the Khovanov homology of torus knots, conjectured by Dunfield, Gukov and Rasmussen in [11].

Theorem 26: *There exists stable Khovanov homology for torus knots.*

An analogous categorification of the n-specializations of the HOMFLYPT polynomial was carried out by M. Khovanov and L. Rozansky in 2004 ([30]). For each positive integer n and link L in S^3 they defined a graded chain complex, with grading preserving differentials, whose graded Euler characteristic is equal to the n-specialization of the HOMFLYPT polynomial of the link L. The construction uses the state-sum model for the HOMFLYPT polynomial ([38]) and is analogous to the categorification of the Jones polynomial: it uses the same cubic complex construction, and there exists a similar long exact sequence in homology. However, since the state-sum model for the HOMFLYPT polynomial is much more complicated than Kauffman's state-sum model, the explicit calculation of the homology groups is practically impossible. Consequently, the values of the $sl(n)$-link homology are known only for a very small class of knots – the two-bridge knots (see [44]).

On the other hand, our approach to the properties of Khovanov homology for positive braid knots and torus knots, essentially relies on the form of the construction (cubic complex) and on the existence of a long exact sequence in homology. Hence, with some minor modifications of the proofs in the $sl(2)$ case we obtain.

Theorem 19 (Chapter 5): *The first $sl(n)$-homology group of a posi-*

tive braid knot is trivial for every positive integer n.

Theorem 27 (Chapter 6): *There exists stable $sl(n)$-homology for torus knots.*

1.3 Graph homology

Apart from the categorifications of the Jones and of the HOMFLYPT polynomial mentioned in the previous section, there also exist categorifications of various other polynomial link invariants, see [27], [25], [6]. All these constructions start from a state-sum expression of the polynomial invariant, and then "lift" it to a chain complex.

Since the chromatic polynomial for graphs, [8], also possess a state-sum expression, it means that one can try to categorify it as well. This was done by L. Helme-Guizon and Y. Rong in 2004, [15]. For each graph G, they defined a graded chain complex whose graded Euler characteristic is equal to the chromatic polynomial of G. Since for graphs there is no isotopy invariance, and therefore no analogs of Reidemeister moves, the homology groups are automatically invariants. Hence, there are no restrictions on the algebraic structure in the categorification of graph invariants and consequently L. Helme-Guizon and Y. Rong, together with J. Przytycki have generalized their construction for an arbitrary algebra, [16], [17]. On the other hand, since the structure of the homology for graphs is simpler than that for links, it can be seen as a step towards a better understanding of link homology ([17], [40]).

A natural generalization of the chromatic polynomial, is the two-variable dichromatic polynomial (see [22]), which is up to a change of variables equal to the important Tutte polynomial of graphs ([8], [19]). Since the dichromatic polynomial is a two-variable polynomial which contains the chromatic polynomial as an one-variable specialization, we have a similar situation to that for the HOMFLYPT polynomial and the Jones polynomial.

In this thesis we categorify the dichromatic polynomial (and consequently the Tutte polynomial). In Chapter 3, we define an infinite set of one-variable specializations of the dichromatic polynomial and then we categorify each of these polynomials. We also give an explicit computation of the homology groups of polygon graphs, where we use the known relation between graph homology and Hochschild homology, see [40].

To each planar graph G there corresponds bijectively, up to mirror image, an alternating link L^G. Then the Jones polynomial of an alternating link

L^G corresponds to a certain one-variable specialization of the dichromatic polynomial, $J_G(q)$, of an appropriate planar graph G (see Appendix A). However, the set of one-variable specializations that we defined previously "misses" the polynomial $J_G(q)$, so we also define a second set of one-variable specializations of the dichromatic polynomial that contains $J_G(q)$, and then categorify each of these polynomials.

In 2005 there was a new development, when M. Khovanov and L. Rozansky in [31] defined triply-graded link homology, i.e. to each link L they associated a doubly-graded chain complex, whose doubly-graded Euler characteristic is equal to the full HOMFLYPT polynomial of L. The method used in the construction, i.e. a cubic complex, is the same as in all previous categorifications – the only difference is that in the explicit definition of chain groups they use Koszul complexes (Koszul complexes are also used in [30] in the categorification of the n-specializations of the HOMFLYPT polynomial). In Section 4.1 of Chapter 4 we give a slightly simplified version of this construction. Then, in Section 4.2 of Chapter 4 we categorify the whole two-variable dichromatic polynomial for graphs. Specifically, for each graph G we construct a doubly-graded chain complex whose doubly-graded Euler characteristic is equal to the dichromatic polynomial. The construction is in the spirit of the construction of the triply-graded theory for links, described in Section 4.1.

1.4 Organization of the thesis

The organization of the thesis is as follows: in Chapter 2 we establish the notation that we will use throughout the thesis and recall briefly the basic definitions (Jones and HOMFLYPT polynomial and $sl(2)$ and $sl(n)$ link homology).

In Chapter 3 we define two different sets of one-variable specializations of the dichromatic polynomial, and then categorify each of the specializations. Also we present some calculations for polygon graphs and give an explicit categorification of the analog of the Jones polynomial of the corresponding alternating link.

In the first section of Chapter 4 we give a simplified definition of the triply-graded homology theory for links. Then, in the second section of the same chapter we define an analogous triply-graded theory for graphs, i.e. we define a doubly-graded chain complex whose doubly-graded Euler char-

acteristic is equal to the two-variable dichromatic polynomial.

In the next two chapters we deal with the properties of link homology. In Chapter 5 we prove that the first (Khovanov) homology of an arbitrary positive braid knot is trivial. We also extend the proof to show that the first $sl(n)$ homology group of a positive braid knot is trivial for every positive integer n.

Finally, in Chapter 6 we prove that all non-alternating torus knots are homologically thick. In the course of the proof we also obtain a stability property of Khovanov homology for torus knots and explicit values for the homology of torus knots for low homological degrees. Also, we show that an analogous stability property holds for the $sl(n)$ homology of torus knots.

At the end of the thesis we have added two Appendices, since we wanted the thesis to be as self-contained as possible. In the first one we explain the relation between planar graphs and alternating links, as well as between the Jones polynomial of an alternating link and a certain specialization of the dichromatic polynomial of the corresponding graph. In Appendix B we discuss the most general diagrammatics based on the state-sum relation and we show how the models of the HOMFLYPT polynomial of Sections 2.5 and 4.1 naturally arise.

Most of this thesis, specifically Chapters 3-6, is based on papers by the author, see [48], [49], [50], [51].

Chapter 2

Notation and Definitions

2.1 Knots and links

By a knot we mean a smoothly embedded circle S^1 in 3-space $\mathbb{R}^3$ (or S^3). By a link we mean a finite disjoint collection of knots. We will identify isotopic knots (and links), i.e. we will identify those that can be continuously transformed into each other. Every knot that is isotopic to the standard circle S^1 we will call the unknot.

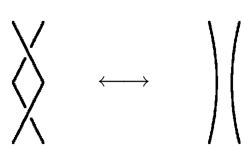
By a knot invariant we mean any function on the set of knots which has the same value on isotopic knots, and analogously for a link invariant.

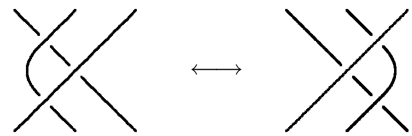
To every link L we can associate (infinitely many, of course) planar projections D, which are also called regular diagrams (or just diagrams). In fact, regular diagrams are planar 4-valent graphs with slightly modified vertices such that one can distinguish between the over- and undercrossing. These modified vertices we call the crossings of the diagrams D and we denote the set of all crossings of D by $c(D)$.

The famous theorem of Reidemeister [45], states that two diagrams D_1 and D_2 represent two isotopic links if and only if one can be transformed into the other by a finite sequence of Reidemeister moves R1, R2 and R3:

R1:

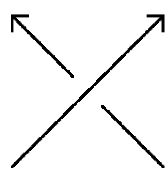
R2:



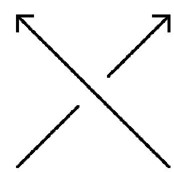
R3: 

Here we mean that two diagrams are related by the Reidemeister move if they look the same except locally where they are like the left and right picture, respectively, of any of the above moves. Hence, a function defined on the set of planar diagrams will be a link invariant if and only if it is invariant under the Reidemeister moves.

We can also introduce an orientation on knots and links. Then locally each crossing of D will look like one of the following two pictures



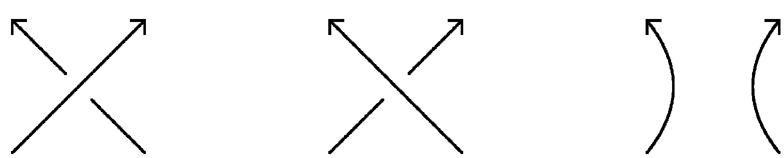
positive



negative

If it looks like the picture on the left we call it a positive crossing and if it looks like the picture on the right we call it a negative crossing. The number of positive (respectively negative) crossings of the diagram D we denote by $n_+(D)$ (respectively $n_-(D)$). We say that a link is positive if it has a planar projection whose crossings are all positive.

In this thesis we will deal with knot invariants, I, that satisfy a skein relation. A skein relation is a linear relation between the values of the invariant I on three oriented links L_+, L_- and L_0 which have identical planar projections except near one crossing where they are as in the following picture:



L_+ $\qquad$ L_- $\qquad$ L_0

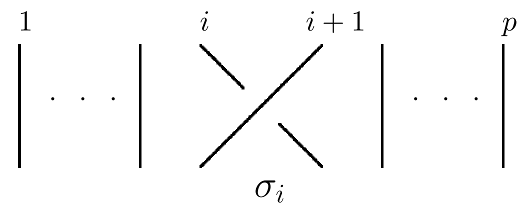
2.2 Braids

Braids are a special type of open links. An n strand braid is obtained by fixing n points in the plane and the same n points in a parallel copy of the

same plane below the first one. A braid is obtained by attaching the top n points to the bottom n points by strings (later called strands) in such a way that the strings never intersect each other and such that the tangent vector to a string is never parallel to the planes. We identify braids up to isotopy: we only assume that the $2n$ endpoints are fixed and the tangent vectors satisfy the above properties through the isotopy.

The most useful thing about braids is that they form groups under obvious concatenation. We denote the group of n-strand braids by B_n.

The braid group may be converted into a purely algebraic object by means of the Artin presentation (see [3]). The generators of the group B_n are $\sigma_1, \ldots, \sigma_{n-1}$ where σ_i is:

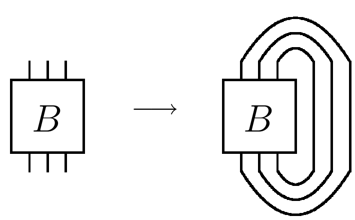


Then the presentation of the braid group is given by the relations

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \end{aligned}$$

We will identify the braid itself and the braid word (word in the alphabet $\sigma_1, \sigma_1^{-1}, \ldots, \sigma_{n-1}, \sigma_{n-1}^{-1}$) that represents it.

We will assume that all strands of a braid are oriented from bottom to top. From any braid, B, we can obtain a knot (or link) by closing the strands i.e. by connecting the initial points to the endpoints by means of a collection of parallel strands:



Conversely, it was shown by Alexander that every link or knot can be put in the form of a closed braid, i.e. it is isotopic to a link which is the closure of a braid ([2]). We say that a knot (or link) is a positive braid knot if it has a planar projection which is the closure of a positive braid, i.e. a braid whose crossings are all positive.

2.3 The Jones polynomial

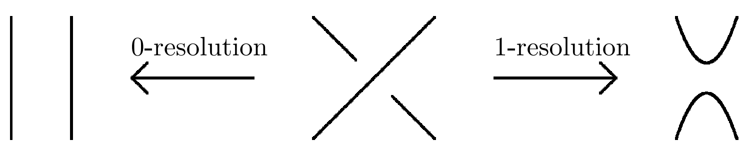
In this section we recall the definition of the Jones polynomial. It was defined by V.F.R. Jones in 1985 in [20]. He obtained the polynomial by introducing a certain tower of von Neumann algebras and by constructing a certain representation into the Artin braid group. The Jones polynomial $J(L)$ is a Laurent polynomial with values in $\mathbb{Z}[t^{1/2}, t^{-1/2}]$ satisfying the following skein relation

$$tJ(L_+) - t^{-1}J(L_-) = (t^{1/2} - t^{-1/2})J(L_0), \tag{2.1}$$

and such that the value of the unknot is 1.

In 1987, L. Kauffman in [23] introduced a state-sum model of the link L, and obtained an alternative definition of the Jones polynomial, which is remarkably simple and purely combinatorial. We will describe Kauffman's state-sum model. In order to do that, we will use slightly different variables than the original ones of Kauffman. Specifically, we will use the ones introduced by M.Khovanov in [24] which are now widely used in link homology theory.

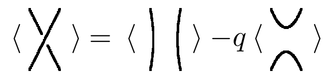
Let L be a link, and let D be a planar projection of L. Then for each crossing $c \in D$ we will define two resolutions (or states) of c according to the following picture:



We call a collection of circles that are obtained by resolving all crossings of D, a total resolution of the diagram D. We order the set of crossings of D, say from 1 to n, where n is the number of crossings of D. Then, obviously, we have a bijection between the set of the total resolutions of the diagram D and the set $\{0,1\}^n$, i.e. the set of all n-tuples of zeros and ones. To each element $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n$ we assign the resolution D_ϵ which is obtained by resolving the i-th crossing as an ϵ_i-resolution. We denote by $|\epsilon|$ the sum of elements of ϵ, and by $c(\epsilon)$ the number of disjoint circles of the resolution D_ϵ. Now we define the Kauffman bracket of the diagram D by:

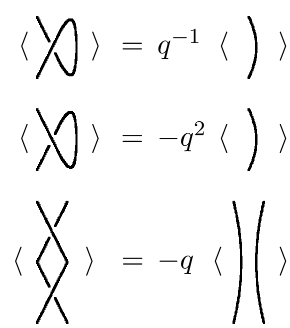
$$\langle D \rangle = \sum_{\epsilon \in \{0,1\}^n} (-1)^{|\epsilon|} q^{|\epsilon|} (q + q^{-1})^{c(\epsilon)}. \tag{2.2}$$

Alternatively, we can describe the Kauffman bracket by the following recursive axioms



$$\langle U_k \rangle = (q + q^{-1})^k,$$

where U_k is the collection of k disjoint circles in the plane. Here in the first formula we assume that the relation is valid for any three diagrams that are the same except near one crossing where they look like the picture. It can be easily verified that the Kauffman bracket is invariant under the R3 move, and that under the R1 and R2 moves it has the following simple behaviour



Then the polynomial $\hat{J}(D)$ defined by

$$\hat{J}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle,$$

is obviously invariant under all Reidemeister moves. Hence, we can write $\hat{J}(L)$, and we call it the unnormalized Jones polynomial. From this definition, we easily obtain the skein relation:

$$q^{-2}\hat{J}(L_+) - q^2 \hat{J}(L_-) = (q^{-1} - q)\hat{J}(L_0),$$

and the value of the unknot is $q + q^{-1}$. Thus, if we define $J(L) = \hat{J}(L)/(q + q^{-1})$ then $J(L)$ is the Jones polynomial (with the variable t from (2.1) replaced by q^{-2}).

2.4 Khovanov homology ($sl(2)$ link homology)

Here we recall the definition of Khovanov homology for links (see [5],[24]). The idea of Khovanov homology is to define a graded chain complex whose graded Euler characteristic is equal to the Jones polynomial. This is done

by following the lines of the Kauffman bracket construction and by "categorifying" the polynomials involved, i.e. by replacing the polynomials with graded modules (whose graded dimensions are equal to the polynomials), sums with direct sums, products with tensor products, etc. We first recall the basic properties of graded $\mathbb{Z}$-modules:

2.4.1 Graded dimension of a graded $\mathbb{Z}$-module

Definition 1 *Let $M = \oplus_k M_k$ for $k \in \mathbb{Z}$ be a graded $\mathbb{Z}$-module where $\{M_k\}$ denotes the k-th graded component of M. The graded (or quantum) dimension of M is the power series*

$$q\dim M := \sum_k q^k \operatorname{rank}(M_k),$$

where $\operatorname{rank}(M_k) = \dim_{\mathbb{Q}}(M_k \otimes \mathbb{Q})$.

The direct sum and tensor product can be defined in the graded category in an obvious way. The following proposition is straightforward.

Proposition 1 *Let M and N be graded $\mathbb{Z}$-modules. Then*

$$\begin{aligned} q\dim(M \oplus N) &= q\dim(M) + q\dim(N) \\ q\dim(M \otimes N) &= q\dim(M) q\dim(N). \end{aligned}$$

Definition 2 *Let $\{l\}$, $l \in \mathbb{Z}$, be the "degree shift" operation on graded $\mathbb{Z}$-modules. That is, if $M = \oplus_k M_k$ is a graded $\mathbb{Z}$-module where M_k denotes the k-th graded component of M, we set $M\{l\}_k := M_{k-l}$. Then we have $q\dim M\{l\} = q^l q\dim M$. In other words, all the degrees are increased by l.*

Now go back to the construction of Khovanov homology. Let L be a link, and D its planar projection. Take an ordering of the crossings of D. Denote by n the number of crossings of D. Then, as before, there is a bijective correspondence between the total resolutions of D and the set $\{0,1\}^n$.

Now we organize the Kauffman bracket state-sum expression (2.2) in the following way:

$$\langle D \rangle = \sum_{i=0}^{n} (-1)^i q^i \sum_{|\epsilon|=i} (q+q^{-1})^{c(\epsilon)}. \tag{2.3}$$

To each total resolution D_ϵ we want to assign a graded module whose quantum dimension is equal to $q^{|\epsilon|}(q+q^{-1})^{c(\epsilon)}$. Let V be a graded $\mathbb{Z}$-module V, which is freely generated by two basis vectors 1 and X, with $\deg 1 = 1$

and $\deg X = -1$. Then to D_ϵ we assign a graded module V_ϵ, which is the tensor product of V's over all circles in the resolution. We draw all the resolutions as the vertices of a (skewed) n-dimensional cube such that in the i-th column we have the resolutions D_ϵ with $|\epsilon| = i$. We define the i-th chain group $C^i(D)$ by:

$$C^i(D) = \oplus_{|\epsilon|=i} V_\epsilon\{i\}.$$

We obviously have that the quantum dimension of $C^i(D)$ is equal to $q^i \sum_{|\epsilon|=i} (q+q^{-1})^{c(\epsilon)}$.

Now, we want to obtain the whole graded chain complex, i.e. we want to define the differential. First, we give the general definitions for the graded chain complexes.

2.4.2 Graded chain complexes and the graded Euler characteristic

Definition 3 *Let $M = \oplus_j M_j$ and $N = \oplus_j N_j$ be graded $\mathbb{Z}$-modules where M_j and N_j denote the j-th graded component of M and N, respectively. A $\mathbb{Z}$-module map $\alpha : M \to N$ is said to be graded with degree d if $\alpha(M_j) \subset N_{j+d}$, i.e. elements of degree j are mapped to elements of degree $j+d$. A $\mathbb{Z}$-module map is called degree preserving (or grading preserving) if it is graded of degree zero.*
A graded chain complex is a chain complex for which the chain groups are graded $\mathbb{Z}$-modules and the differentials are graded.

Definition 4 *Let $[s]$, $s \in \mathbb{Z}$, be the "height shift" operation on chain complexes. That is, if C is a chain complex, $\cdots \to C^r \to C^{r+1} \cdots$ of modules (or vector spaces), and if $\bar{C} = C[s]$, then $\bar{C}^r = C^{r-s}$, with all differentials shifted accordingly.*

Definition 5 *The graded Euler characteristic $\chi_q(C)$ of a graded chain complex C is the alternating sum of the graded dimensions of its homology groups, i.e. $\chi_q(C) = \sum_i (-1)^i q\dim(H^i)$.*

Proposition 2 *([5], [15]) If the differentials are degree preserving and all chain groups are finite-dimensional, the graded Euler characteristic is also equal to the alternating sum of the graded dimensions of its chain groups, i.e.*

$$\chi_q(C) = \sum_i (-1)^i q\dim(H^i) = \sum_i (-1)^i q\dim(C^i).$$

Remark 1 *From the last Proposition we have that if we define a degree preserving differential between the groups $C^i(D)$, the chain complex $C(D)$ obtained in this way will have as Euler characteristic the Kauffman bracket $\langle D\rangle$. Hence, we are left with defining such a differential.*

The differentials $d^i : C^i(D) \to C^{i+1}(D)$ are defined as the (signed) sum of "per-edge" differentials. Specifically, the only nonzero maps are from D_ϵ to $D_{\epsilon'}$, where $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_i \in \{0, 1\}$, if and only if ϵ' has all entries the same as ϵ except one ϵ_j, for some $j = 1, \ldots, m$, which is changed from 0 to 1. We denote these differentials by $d_\nu : V_\epsilon\{|\epsilon|\} \to V_{\epsilon'}\{|\epsilon'|\}$, where ν is the n-tuple which consists of the label $*$ at the position j and of $n-1$ 0's and 1's (the same as the remaining entries of ϵ).

Note that in these cases, either two circles of D_ϵ merge into one circle of $D_{\epsilon'}$ or one circle of D_ϵ splits into two circles of $D_{\epsilon'}$, while all other circles remain the same.

In the first case, we define the map d_ν as the identity on the tensor factors (V) that correspond to the unchanged circles, and on the remaining factors we define it to be the (grading preserving) multiplication map $m : V \otimes V \to V\{1\}$, which is given on basis vectors by:

$$m(1 \otimes 1) = 1, \quad m(1 \otimes X) = m(X \otimes 1) = X, \quad m(X \otimes X) = 0.$$

In the second case, we define the map d_ν as the identity on the tensor factors (V) that correspond to the unchanged circles, and on the remaining factors we define it to be the (grading preserving) comultiplication map $\Delta : V \to V \otimes V\{1\}$, which is given on basis vectors by:

$$\Delta(1) = 1 \otimes X + X \otimes 1, \quad \Delta(X) = X \otimes X.$$

Finally, to obtain the differential d^i of the chain complex $C(D)$, we sum all contributions d_ν with $|\nu| = i$, multiplied by the sign $(-1)^{f(\nu)}$, where $f(\nu)$ is equal to the number of 1's ordered before $*$ in ν. This makes every square of our cubic complex anticommutative, and hence we obtain a genuine differential (i.e. $d^{i+1}d^i = 0$).

We denote the homology groups of the complex $(C(D), d)$ obtained in this way by $H^i(D)$ and call them the *unnormalized homology groups of D*. Since the chain groups are naturally graded by the construction of the chain complex, so are the homology groups, and we write $H^{i,j}(D)$ for the summand of degree j in $H^i(D)$. In order to obtain link invariants (i.e. independence of the chosen projection), we have to shift the chain complex (and hence the homology groups) by:

$$\mathcal{C}(D) = C(D)[-n_-]\{n_+ - 2n_-\}, \tag{2.4}$$

where n_+ and n_- are the number of positive and negative crossings, respectively, of the diagram D. The homology groups of the shifted complex $\mathcal{C}(D)$ we denote by $\mathcal{H}^i(D)$. Hence, we have $\mathcal{H}^{i,j}(D) = H^{i+n_-,j-n_++2n_-}(D)$.

Theorem 2 *([24],[5]) The homology groups $\mathcal{H}^i(D)$ are independent of the choice of the planar projection D. Furthermore, the graded Euler characteristic of the complex $\mathcal{C}(D)$ is equal to the Jones polynomial of the link L.*

Hence, we can write $\mathcal{H}(L)$, and we call $\mathcal{H}^i(L)$ *the homology groups of the link L*.

2.4.3 Basic properties of Khovanov homology

If the diagram D has only positive crossings then we do not have an overall shift in the homology degrees, and so we have that, for example, $\mathcal{H}^1(L)$ is trivial if and only if $H^1(D)$ is trivial. Also, in the general case if we have a positive knot K (a knot that has a planar projection with only positive crossings) then we have that $\mathcal{H}^i(K)$ is trivial for all $i < 0$. Furthermore, if D is a planar projection of a positive knot K with n_- negative crossings then $H^i(D)$ is trivial for $i < n_-$.

Usually, the homology groups of the link K are represented as a planar array in such a way that $\operatorname{rank}\mathcal{H}^{i,j}(K) = \dim(\mathcal{H}^{i,j}(K) \otimes \mathbb{Q})$ (or the whole group $\mathcal{H}^{i,j}(K)$, if we want to keep track of the torsions) is specified in the position (i, j). As can be easily seen, the q-gradings (j) of the generators of nontrivial $\mathcal{H}^{i,j}(K)$ and the number of components of K are of the same parity (either all are even or all are odd). Hence, by a *diagonal* of the homology of the link K, we mean a line $j-2i = a = \mathit{const}$, when there exist integers i and j such that $j - 2i = a$ and $\operatorname{rank}\mathcal{H}^{i,j}(K) > 0$. If $a_{\max}$ and $a_{\min}$ are the maximal and minimal value of a such that the line $j - 2i = a$ is a diagonal of the homology of the link K, then we define the homological width of the link K to be $h(K) = (a_{\max} - a_{\min})/2 + 1$.

Every knot (link) occupies at least two diagonals (i.e. $h(K) \geq 2$ for every link K), and the ones that occupy exactly two diagonals are called H-thin, or homologically thin. For example all alternating knots are H-thin ([33]), and the free part of the homology of any H-thin knot is determined by its Jones polynomial and the signature. A knot that is not H-thin is called H-thick or homologically thick.

Furthermore, for an element $x \in \mathcal{H}^{i,j}(K)$, we denote its homological grading – i – by $t(x)$, and its q-grading (also called the quantum grading) –

j – by $q(x)$. We also introduce a third grading $\delta(x)$ by $\delta(x) = q(x) - 2t(x)$. Hence, we have that the knot K is H-thick, if there exist three generators of $\mathcal{H}(K)$ with different values of the δ-grading.

An alternative way of presenting the homology of the knot is by means of the two-variable Poincaré polynomial $P(K)(t,q)$ of the chain complex $\mathcal{C}(D)$, i.e.:

$$P(K)(t,q) = \sum_{i,j\in\mathbb{Z}} t^i q^j \operatorname{rank} \mathcal{H}^{i,j}(K).$$

Let D be a diagram of a link L and let c be one of its crossings. Denote by D_i, $i = 0,1$ the diagram that is obtained after performing an i-resolution of the crossing c. Then there exists a long exact sequence of (unnormalized) homology groups (see e.g. [57] and Remark 13 in Chapter 3):

$$\cdots \to H^{i-1,j-1}(D_1) \to H^{i,j}(D) \to H^{i,j}(D_0) \to H^{i,j-1}(D_1) \to H^{i+1,j}(D) \to \cdots \tag{2.5}$$

2.5 The HOMFLYPT polynomial

The HOMFLYPT polynomial $P(L)(a,q)$, named after its many discoverers ([12], [41]), is a two-variable generalization of the Jones polynomial. It is given by the following skein relation:

$$aP(L_+) - a^{-1}P(L_-) = (q - q^{-1})P(L_0), \tag{2.6}$$

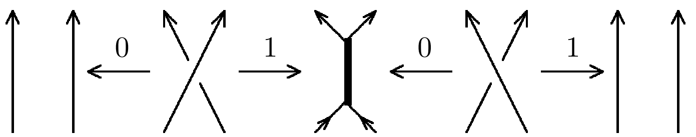
and by prescribing the value of the unknot to be equal to 1. As can be easily shown ([39]), the skein relation (2.6) is the most general skein relation of this form.

There is an alternative description of the HOMFLYPT polynomial, given by defining an infinite set of one-variable specializations, by $P_n(L)(q) = P(L)(q^n, q)$, one for each positive integer n, which is enough to recover the whole HOMFLYPT polynomial. Obviously, the case $n = 2$, i.e. the polynomial $P_2(L)(q)$ is equal to the Jones polynomial $J(L)(q^{-1})$.

2.5.1 A state-sum model

In order to categorify the HOMFLYPT polynomial, we need an analogous state-sum model for its n-specializations so that we can proceed as in the case of $sl(2)$-link homology. Such a model was introduced by Murakami, Ohtsuki and Yamada in [38] (in greater generality), and also previously by Kauffman ([22], section I.11).

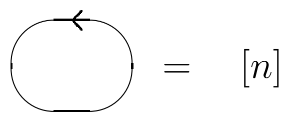
Let L be an oriented link, and let D be a planar projection of L. Then to each crossing c of D we associate 0- and 1-resolutions, depending on the sign of the crossing c, according to the following picture:

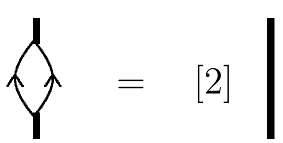


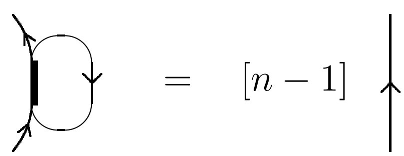
(2.7)

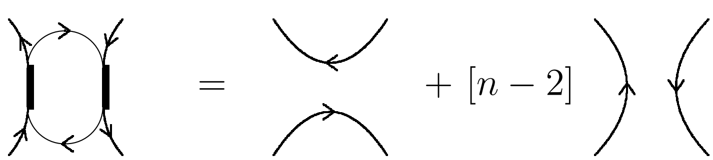
Again we have a bijection between the set of all n-tuples ϵ of 0's and 1's and the set of total resolutions D_ϵ. The total resolutions in this case consist of the planar trivalent graphs Γ, such that at each vertex we have exactly one (unoriented) wide edge and two oriented thin edges, and such that at one end of each wide edge both thin edges are incoming and at the other end both thin edges are outgoing.

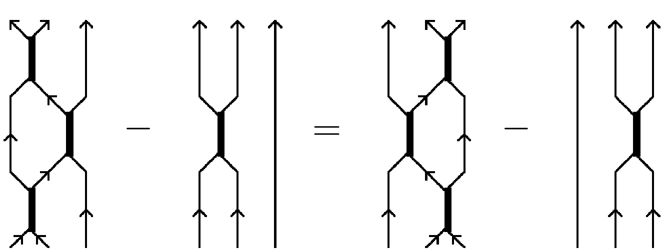
Now, introduce the function $\hat{P}_n(\Gamma)(q) \in \mathbb{Z}[q, q^{-1}]$ on the set of such planar trivalent graphs, by the following five axioms:







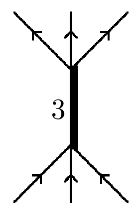




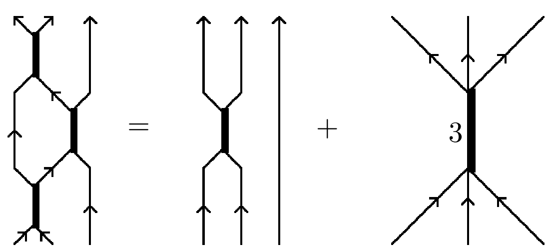
Again by the above relations we mean the respective equality of the value of the function $\hat{P}_n$ on the trivalent graphs which look the same except one small part where they are like the left and right side, respectively, of the above equalities. We denote by $[k]$ the quantum integer k, given by the expression:

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}} = q^{k-1} + q^{k-3} + \ldots + q^{1-k}.$$

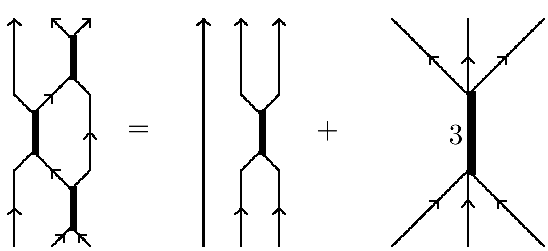
The consistency of the above axioms is shown in [38]. The last relation can be rewritten by introducing four-valent vertices and a new type of (thick) edges, that we label with the number 3:



and we define the value of $\hat{P}_n$ on this graph as equal to the expression on either side of the last of the above axioms. In other words, we can replace the fifth axiom by the following two relations (see [38] and Section 1 of [30]):



(2.8)



(2.9)

Also, from [38] we know that the value of $\hat{P}_n(\Gamma)$ on every trivalent graph Γ with thick edges is a Laurent polynomial in q and q^{-1} whose coefficients

are nonnegative integers.

Like in the case of the Jones polynomial, we will extend the definition of the function $\hat{P}_n(\Gamma)$ to oriented links by the following recursive relations:

$$\hat{P}_n(D_+) = q^{1-n}\hat{P}_n(D_0) - q^{-n}\hat{P}_n(D_\Gamma) \tag{2.10}$$
$$\hat{P}_n(D_-) = q^{n-1}\hat{P}_n(D_0) - q^{n}\hat{P}_n(D_\Gamma). \tag{2.11}$$

Here we denote by D_+, D_-, D_0 and D_Γ any four diagrams that look the same except near one crossing where they look like:

D_+ D_- D_0 D_Γ

Then it can be easily verified that $\hat{P}_n(D)$ is invariant under the Reidemeister moves, and hence that it is a link invariant. Thus, we can write $\hat{P}_n(L)$. Furthermore, from (2.10) and (2.11) we obtain the following skein relation for $\hat{P}_n$:

$$q^n\hat{P}_n(L_+) - q^{-n}\hat{P}_n(L_-) = (q-q^{-1})\hat{P}_n(L_0).$$

Hence, if we define $P_n(L) = \hat{P}_n(L)/[n]$, we obtain the n-specialization of the HOMFLYPT polynomial.

2.6 Khovanov-Rozansky homology ($sl(n)$ link homology)

2.6.1 The construction of the chain complex

The categorification of the $sl(n)$ link invariant, analogous to the categorification of the Jones polynomial, was defined by Khovanov and Rozansky in [30]. The construction follows the same lines as the cubic complex construction of Section 2.4, and uses the state-sum model for the $sl(n)$ link invariant from the previous section. The main difference is that the contributions from the states (i.e. the total resolutions) given by $\hat{P}_n(\Gamma)$ are much more complicated expressions (so that they satisfy the five axioms from the previous section) than in the Jones polynomial case (where the contributions are just $(q+q^{-1})^k$). Fortunately, because of the positivity of the calculus (the coefficients of $\hat{P}_n(\Gamma)$ are nonnegative integers), it was possible to assign graded vector space to any state Γ, with graded dimension equal to $\hat{P}_n(\Gamma)$

(see below).

Again for every regular diagram D, they also defined a graded chain complex whose graded Euler characteristic is equal to the unnormalized $sl(n)$ link invariant - $\hat{P}_n(D)$. However, we will use a slightly different, intermediate, normalization $Q_n(D)$ instead of $\hat{P}_n(D)$ from the previous section. On the level of trivalent graphs Q_n and $\hat{P}_n$ coincide. However, for the recursive relations we take:

$$\begin{aligned} Q_n(D_+) &= Q_n(D_0) - q^{-1}Q_n(D_\Gamma) && (2.12)\\ Q_n(D_-) &= Q_n(D_\Gamma) - q^{-1}Q_n(D_0). && (2.13)\end{aligned}$$

As a result, we obviously have $\hat{P}_n(D) = (-1)^{-n_-}q^{-(n-1)\cdot n_+ + n\cdot n_-}Q_n(D)$.

To every total resolution D_ϵ they assigned a graded $\mathbb{Q}$-vector space $\bar{V}_\epsilon$ which is the cohomology of a certain 2-periodic complex $C(D_\epsilon)$ (for details see [30] and the following subsection). It can be shown that $q\dim \bar{V}_\epsilon = \hat{P}_n(D_\epsilon) = Q_n(D_\epsilon)$. Indeed, this is proved by showing that the complex $C(D_\epsilon)$ categorifies the axioms for $\hat{P}_n(D_\epsilon)$, in the sense that, for example, the second axiom becomes:

$$C(\Gamma_1) \cong C(\Gamma_2)\{1\} \oplus C(\Gamma_2)\{-1\},$$

where by Γ_1 and Γ_2 we have denoted the graphs on the left and on the righthand side, respectively, of the second axiom.

Again, the resolutions are organized in the same cubic complex as in Section 2.4 (just bearing in mind the different 0- and 1-resolutions, see (2.7)) and we obtain the chain groups by summing along columns:

$$C_n^i(D) = \bigoplus_{|\epsilon|=i} \bar{V}_\epsilon\{-i\}. \qquad (2.14)$$

Furthermore, we define the grading-preserving differential $d_n^i : C_n^i(D) \to C_n^{i+1}(D)$ as a signed sum of per-edge maps (the signs are the same as in the $sl(2)$-case, Section 2.4). For the explicit definition of the per-edge maps see [30] and the following subsection. For our purposes, we want to point out that the complex associated to a (generalized) diagram D, with a specified crossing c, is again the mapping cone of a certain homomorphism $f : C_n(D_0) \to C_n(D_1)\{-1\}$, where by D_i, $i = 0, 1$, we have denoted the diagram obtained from D by performing the i-resolution of the crossing c (according to (2.7)). Furthermore, a complex $C_n(D')$ can be associated to any generalized diagram D', i.e. a regular diagram with thick edges.

We define the homology groups of the complex $(C_n(D), d_n)$ by $H_n^i(D) =$

$H^i(C_n(D))$. Like in the $sl(2)$ case, the homology groups $H_n^i(D)$ inherit a natural grading from the internal gradings of the chain groups and we write $H_n^{i,j}(D)$ for the subset of elements of $H_n^i(D)$ of the degree j.

Finally we define a graded chain complex $\mathcal{C}_n(D)$ by

$$\mathcal{C}_n(D) = C_n(D)[-n_-]\{(1-n)n_+ + n \cdot n_-\},$$

where n_- and n_+ are the numbers of negative and positive crossings of D, respectively.

Since the differential is grading preserving we have that the graded Euler characteristic of the chain complex $\mathcal{C}_n(D)$ is equal to $\hat{P}_n(D)$. In fact, we have

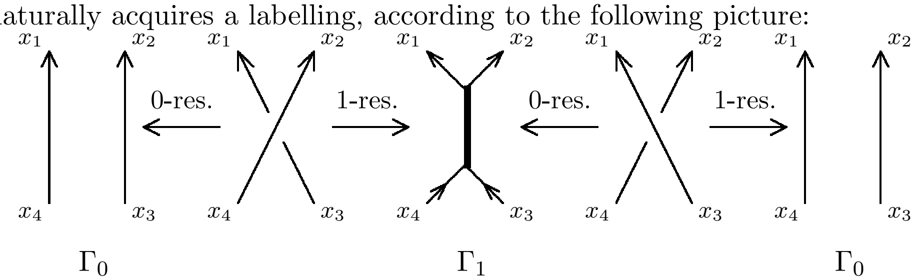
Theorem 3 *[30] The homology groups $\mathcal{H}_n^i(D) = H^i(\mathcal{C}_n(D))$ are independent of the planar projection D of the link L. Hence, the homology groups are link invariants and we can write $\mathcal{H}_n^i(L)$.*

Remark 4 *If we put $n = 2$ in the $sl(n)$ theory [30] we obtain the standard Khovanov homology [24] with the q-gradings inverted, i.e. we have an isomorphism $\mathcal{H}_2^{i,j}(L) \cong \mathcal{H}^{i,-j}(L) \otimes_{\mathbb{Z}} \mathbb{Q}$. We would obtain the same convention for the standard Khovanov homology (Section 2.4) if we defined* $\deg 1 = -1$, $\deg X = 1$ *and in all later shifts in q-gradings we replaced $\{i\}$ by $\{-i\}$. These conventions for the $sl(2)$ homology theory are used in [26] and [28].*

2.6.2 Some details concerning the construction

As we mentioned above, the explicit construction of the complexes $C(\Gamma)$ and $C_n(D)$ from the previous subsection is rather complicated, and hence we omit the technical details (for these, see [30]). However, we want to point some features of the construction and extract some elementary properties that we will use in the thesis.

In the construction, one starts with a regular diagram D of a link L, and assigns a different variable x_i to every arc between two crossings. We define the ring R as the ring of polynomials over $\mathbb{Q}$ in all variables x_i. We introduce a grading on R by defining $\deg x_i = 2$. Then every resolution naturally acquires a labelling, according to the following picture:



Now, to every such resolution one assigns a certain matrix factorization (i.e. a generalization of a 2-periodic complex – see [30]), and to each total resolution D_ϵ one assigns the tensor product over the crossings of D of the corresponding matrix factorizations. In this way, one obtains a (genuine) 2-periodic complex $C(D_\epsilon)$, whose homology $H(D_\epsilon)$ is the vector space $\bar{V}_\epsilon$ that we assigned to the resolution D_ϵ in the previous subsection. Thus, we are left with defining the particular matrix factorizations.

In the case when we have a thin edge with two variables:

we assign to it the matrix factorization L^i_j, defined by:

$$R \xrightarrow{\pi^n_{i,j}} R\{1-n\} \xrightarrow{x_i - x_j} R$$

where $\pi^n_{i,j}$ is given by:

$$\pi^n_{i,j} = \frac{x_i^{n+1} - x_j^{n+1}}{x_i - x_j} = \sum_{k=0}^{n} x_i^k x_j^{n-k}.$$

Finally, to the resolution Γ_0, we assign the tensor product of matrix factorizations L^1_4 and L^2_3, corresponding to the thin edges that form it.

In the case of the resolution with a thick edge, Γ_1, the corresponding matrix factorization is defined as the tensor product of the factorizations:

$$R \xrightarrow{u_1} R\{1-n\} \xrightarrow{x_1+x_2-x_3-x_4} R$$

and

$$R \xrightarrow{u_2} R\{3-n\} \xrightarrow{x_1x_2-x_3x_4} R$$

with the grading shifted down by 1. Here by u_1 and u_2 we have denoted the following polynomials

$$\begin{aligned} u_1 &= u_1(x_1,x_2,x_3,x_4) = \frac{g(x_1+x_2,x_1x_2) - g(x_3+x_4,x_1x_2)}{x_1+x_2-x_3-x_4}, \\ u_2 &= u_2(x_1,x_2,x_3,x_4) = \frac{g(x_3+x_4,x_1x_2) - g(x_3+x_4,x_3x_4)}{x_1x_2-x_3x_4}, \end{aligned}$$

where $g(x,y)$ is the unique polynomial such that $g(x+y,xy) = x^{n+1}+y^{n+1}$. Although u_1 and u_2 have in general a rather complicated explicit expressions, in the case when $x_1+x_2 = x_3+x_4$ and $x_1x_2 = x_3x_4$, we have that

$u_1 = \pi_{1,2}^n$ and $u_2 = \pi_{1,2}^{n-1}$.

The main problem with the definitions from above, is the fact that the homology of the complex that corresponds to the total resolution is incalculable in almost all cases. However, we know that the homology of the total complex is an R/I-module, where I is the ideal generated by all polynomials that appear in the definitions of the matrix factorizations (see Proposition 2 of [30] and also Section 2.3 of [14]). Furthermore, in some very simple cases like all the total resolutions D_ϵ of a standard diagram D of a torus $(2, n)$-knot (see Chapters 5 and 6), the homology $H(D_\epsilon)$ is isomorphic to the above quotient ring of polynomials (up to a certain shift in grading).

Now, let us move on to the differential. As can be shown, the homology $H(D_\epsilon)$ is concentrated in only one homological degree of the 2-periodic complex $C(D_\epsilon)$ (the one with parity equal to the parity of the number of circles in the total resolution of D when we resolve all the crossing into the resolution Γ_0). Since the differential should be a map between two homology groups $H(D_\epsilon)$ and $H(D_{\epsilon'})$ (ϵ and ϵ' are related like in the $sl(2)$ case), we will define it as the map induced on homology by a homomorphism $f_\nu : C(D_\epsilon) \to C(D_{\epsilon'})$, defined as follows:

Obviously, the only nontrivial differentials are between two complexes associated to diagrams D_ϵ and $D_{\epsilon'}$ which look the same except near one crossing where one is of the form Γ_0 and the another one is of the form Γ_1. Since the complex $C(D_\epsilon)$ is defined as the tensor product of the matrix factorizations assigned to Γ_0 and Γ_1, we define the homomorphism f_ν as the tensor product of the identity maps over all crossings where we have the same resolutions, and at the crossing where the resolutions differ, we define it as a homomorphism $\chi_0 : C(\Gamma_0) \to C(\Gamma_1)$ or $\chi_1 : C(\Gamma_1) \to C(\Gamma_0)$. Since the chain groups of the matrix factorizations $C(\Gamma_i)$, $i = 0, 1$, are of rank 2, both maps χ_i can be described by a pair of 2×2 matrices. Explicitly, χ_0 is given by a pair (U_0, U_1) defined by

$$\begin{aligned} U_0 &= \begin{pmatrix} x_1 - x_3 & 0 \\ a_1 & 1 \end{pmatrix}, \\ U_1 &= \begin{pmatrix} x_1 & -x_3 \\ -1 & 1 \end{pmatrix}, \end{aligned}$$

where

$$a_1 = \frac{u_1 + x_1 u_2 - \pi_{2,3}^n}{x_1 - x_4}.$$

The map χ_1 is given by a pair of matrices (V_0, V_1) defined by

$$
\begin{aligned}
V_0 &= \begin{pmatrix} 1 & 0 \\ -a_1 & x_1 - x_3 \end{pmatrix}, \\
V_1 &= \begin{pmatrix} 1 & x_3 \\ 1 & x_1 \end{pmatrix}.
\end{aligned}
$$

It can easily be seen that both maps χ_0 and χ_1 have degree 1. However, the differentials they induce are grading preserving because of the grading shift in the definition of the chain groups (2.14).

Chapter 3

Categorification of the dichromatic polynomial for graphs

3.1 Introduction

In this chapter we categorify the dichromatic polynomial of graphs, and consequently the Tutte polynomial. Since the dichromatic polynomial, $P_G(q,v)$, is a two-variable polynomial and the standard techniques of categorification work with one-variable polynomials, we will define a class of one-variable specializations $P_n(G)$, one for every positive integer n – a family which is enough to recover the original dichromatic polynomial – and then for each of them we construct a chain complex whose graded Euler characteristic is equal to $P_n(G)$.

As is well-known, to each alternating link L there corresponds bijectively a planar graph $G(L)$, and the value of the Jones polynomial of the link L corresponds to the specialization of the Tutte polynomial $T(x,1/x)$ or analogously to the specialization of the dichromatic polynomial $J(q) = P(q,q^2/(q-1))$ (see Appendix A and [8] or [22]). However, the set of specializations from the previous paragraph "misses" the polynomial $J(q)$.

In order to resolve this, we define a different set of one-variable specializations $Q_n(G)$, one for every integer $n \leq 2$, and then categorify each of these one-variable polynomials. Although the chain complexes that categorify this set of specializations have infinite-dimensional chain groups, for $n=2$ we obtain the categorification of the analog $J(q)$ of the Jones polynomial. Furthermore, in the last case ($n=2$) we also give an alternative

description in terms of the enhanced states.

3.2 Preliminaries

Let G be a graph specified by a set of vertices $V(G)$ and a set of edges $E(G)$. If $e \in E(G)$ is an arbitrary edge of the graph G, then by $G - e$ we denote the graph G with the edge e deleted, and by G/e the graph obtained by contracting edge e (i.e. by identifying the vertices incident to e and deleting e). The dichromatic polynomial of the graph, $P_G(q, v)$, is a two-variable generalization of the chromatic polynomial of the graph. It is given by the following axioms:

$$\begin{aligned} (A1) \qquad & P_G = P_{G-e} - qP_{G/e}, \\ (A2) \qquad & P_{N_k} = v^k, \end{aligned}$$

where N_k is the graph with k vertices and no edges.

These two axioms determine the polynomial uniquely. Obviously, if we put $q = 1$, we obtain the usual chromatic polynomial. Furthermore, from $(A1)$ we have a recursive expression for the dichromatic polynomial in terms of the value of the polynomial on graphs with a smaller number of edges. By repeated use of $(A1)$ we will obtain the value of the dichromatic polynomial as a sum of contributions from all spanning subgraphs of G (subgraphs that contain all vertices of G), which we will call states. Furthermore, if for each subset $s \subset E(G)$ we denote by $[G : s]$ the graph whose set of vertices is $V(G)$ and set of edges is s, then the contribution of the graph $[G : s]$ is $(-1)^{|s|}q^{|s|}v^{k(s)}$, where $|s|$ is the number of elements of s and $k(s)$ is the number of connected components of $[G : s]$. Hence, we obtain the expression:

$$P_G(q, v) = \sum_{s \subset E(G)} (-1)^{|s|}q^{|s|}v^{k(s)} = \sum_{i \geq 0} (-1)^i q^i \sum_{s \subset E(G), |s|=i} v^{k(s)},$$

which is called the state-sum expansion of the polynomial P_G.

The important two-variable Tutte polynomial of a graph G, $T_G(x, y)$ is given by a similar recursive relation to the one for the dichromatic polynomial. Namely, it is given by

$$T_G(x, y) = \begin{cases} xT_{G/e}(x, y) & \text{if } e \text{ is a bridge} \\ yT_{G-e}(x, y) & \text{if } e \text{ is a loop} \\ T_{G-e}(x, y) + T_{G/e}(x, y) & \text{otherwise} \end{cases}$$

For the Tutte polynomial there exists a state-sum expression given by:

$$T_G(x,y) = \sum_{s\subset E(G)} (x-1)^{k(s)-k(E(G))}(y-1)^{|s|-N+k(s)},$$

where by N we denote the number of vertices of G. It is obviously related to the dichromatic polynomial by:

$$\begin{aligned} T_G(x,y) &= \sum_{s\subset E(G)} (x-1)^{k(s)-k(E(G))}(y-1)^{|s|-N+k(s)} = \\ &= (x-1)^{-k(E(G))}(y-1)^{-N} \sum_{s\subset E(G)} (y-1)^{|s|}((x-1)(y-1))^{k(s)} = \\ &= (x-1)^{-k(E(G))}(y-1)^{-N} P_G(1-y,(x-1)(y-1)). \end{aligned} \tag{3.1}$$

Hence, we have that the Tutte polynomial $T_G(x,y)$ is a multiple of the dichromatic polynomial $P_G(q,v)$, when we take $q = 1-y$ and $v = (x-1)(y-1)$.

Our aim is to define a graded chain complex whose graded Euler characteristic is equal to the dichromatic polynomial. However, since the dichromatic polynomial is a two-variable polynomial, we will first define an infinite set of one-variable specializations and then "categorify" each of the specializations. This situation is very similar to the two-variable HOMFLYPT polynomial for knots and its infinite set of one-variable specializations (one for each positive integer n, with $n = 2$ being the Jones polynomial). Note that in this way we will also categorify the Tutte polynomial.

In Section 3.3, we will observe the first set of specializations of the dichromatic polynomial. Namely, for each positive integer n we define:

$$P_{G,n}(q) = P_G(q^n, 1+q+\ldots+q^n).$$

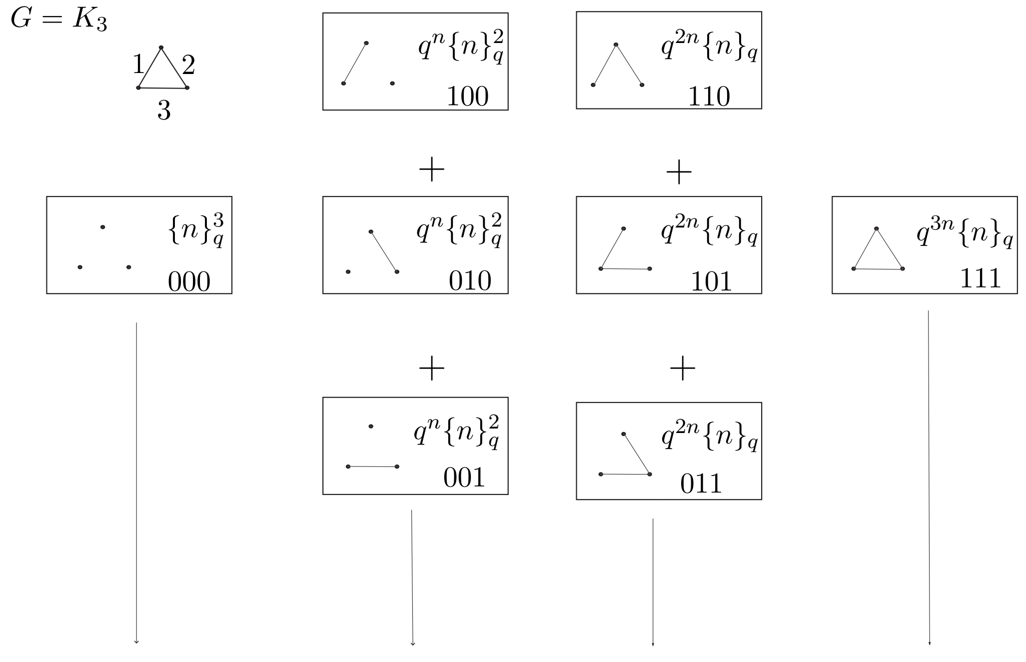
In other words, for each positive integer n, we replaced in the axioms $(A1)$ and $(A2)$ q by q^n, and v by the expression $1+q+\ldots+q^n$, which we will denote by $\{n\}_q$. Hence we have the following state-sum expression for the specialization $P_{G,n}$:

$$P_{G,n}(q) = \sum_{i\geq 0} (-1)^i \sum_{s\subset E(G), |s|=i} q^{n|s|}\{n\}_q^{k(s)}$$

For every positive integer n and graph G, we will define a graded chain complex whose graded Euler characteristic is equal to $P_{G,n}$. Denote by m

the number of edges of the graph G. The construction will depend on the ordering of the edges of G $(e_1, \ldots, e_m)$. Then each subset $s_\epsilon \subset E(G)$ will be uniquely determined by an element $\epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \{0,1\}^m$, where $\epsilon_i = 1$ if $e_i \in s$ and $\epsilon_i = 0$ if $e_i \notin s$. Obviously, each such m-tuple is uniquely determined by the subset s of $E(G)$, and so we have a bijective correspondence between $\{0,1\}^m$ and the set of all $s \subset E(G)$. Thus, every element $\epsilon \in \{0,1\}^m$ determines a set s_ϵ of edges of G, and hence a graph $[G : s_\epsilon]$ which we will denote by G_ϵ.

We will construct our complex by "flattening" the cube of states (see figure 3.1).



$$P_n(G): \quad \{n\}_q^3 \quad - \quad 3q^n\{n\}_q^2 \quad + \quad 3q^{2n}\{n\}_q \quad - \quad q^{3n}\{n\}_q$$

Figure 3.1: State-sum expression for the n-specialization of the dichromatic polynomial

To each vertex ϵ of the cube $\{0,1\}^m$ (which is skewed such that in the i-th column, $0 \le i \le m$, are all the ϵ's such that $|\epsilon| = i$), we will assign a graded $\mathbb{Z}$-module whose graded dimension is equal to the contribution of the subgraph G_ϵ. In other words, a module whose graded dimension is

$q^{n|\epsilon|}\{n\}_q^{k_\epsilon}$, where k_ϵ is the number of connected components of G_ϵ. We will build chain groups by taking direct sums along the columns. Finally, we will define differentials such that they are grading preserving, and in this way we will obtain a chain complex whose graded Euler characteristic is equal to the n-th specialization $P_{G,n}$ of the dichromatic polynomial.

In the next section we will define the chain complex more precisely.

In order to define the second set of specializations of the dichromatic polynomial, we introduce a new variable a as $a = v(q-1)$, i.e. $v = a/(q-1)$. In this way, we obtain a two-variable polynomial $\tilde{P}_G(q,a)$, and the one-variable specializations we will define to be $Q_{G,n}(q) = \tilde{P}_G(q,q^n)$, for every integer $n \leq 2$. Note that we then have $v = q^n/(q-1) = q^{n-1}/(1-q^{-1})$. We will denote the expression $q^n/(q-1) = q^{n-1}\sum_{i\geq 0} q^{-i}$ by q_n.

We also obtain an analogous state-sum expression like for $P_{G,n}$, and then, in Section 3.4, we define a cubic complex along the same lines as outlined before (for the first set of specializations).

3.3 The cubic complex construction of the chain complex

The construction of the complex (for both sets of specializations) will be similar to the case of the Jones polynomial for links (see Section 2.4). So, we will have the same cubic complex and we will use the same definitions of graded dimensions and graded chain complexes as in Section 2.4.

We are going to assign to each state a graded $\mathbb{Z}$-module whose graded dimension is $\{n\}_q^k$ for the first set of specializations, and q_n^k for the second set. The reader should note the following examples of graded modules:

Example 1 *Let V be the graded free $\mathbb{Z}$-module with $n+1$ basis elements: $X^0(=1), X, X^2, \ldots, X^n$ such that the degree of X^i is equal to $n-i$, for $i = 0,\ldots,n$. Then we have $q\dim V = 1+q+\ldots+q^n = \{n\}_q$. Furthermore, we have that $V^{\otimes k}\{l\} = q^l\{n\}_q^k$. Notice that we can also describe V as the quotient of the ring of polynomials by $V = \mathbb{Z}[X]/(X^{n+1})\{n\}$, with $\deg X = -1$.*

Example 2 *Let $M = \mathbb{Z}[x_i]\{n-1\}$ be the ring of polynomials in one variable. If we define the degree of the variable x_i to be -1, then M becomes a graded free $\mathbb{Z}$-module, with quantum dimension $q\dim M = q^{n-1}\sum_{i\geq 0} q^{-i} = q_n$. Furthermore, if $k > 0$, then $M^{\otimes k}$ is isomorphic to the ring of polynomials*

in k variables $M_k = \mathbb{Z}[x_1, \ldots, x_k]\{k(n-1)\}$*, and we have that* $q\dim M^{\otimes k} = q\dim M_k = q_n^k$.

We are now ready to explain our construction. Let n be a positive integer. Let G be a graph with m ordered edges and let V be as in Example 1. To each vertex $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$ of the cube $\{0,1\}^m$, we associate a graded free $\mathbb{Z}$-module V_ϵ given by $V_\epsilon(G) = V^{\otimes k(\epsilon)}\{n|\epsilon|\}$, with V being the graded $\mathbb{Z}$-module defined in Example 1. In other words, we assign one copy of V to each connected component of the resolution of the graph, and then shift the degree up by $n|\epsilon|$. From the definition we have that $q\dim V_\epsilon$ is (up to the sign $(-1)^{|\epsilon|}$) the contribution of P_{G_ϵ} to the dichromatic polynomial.

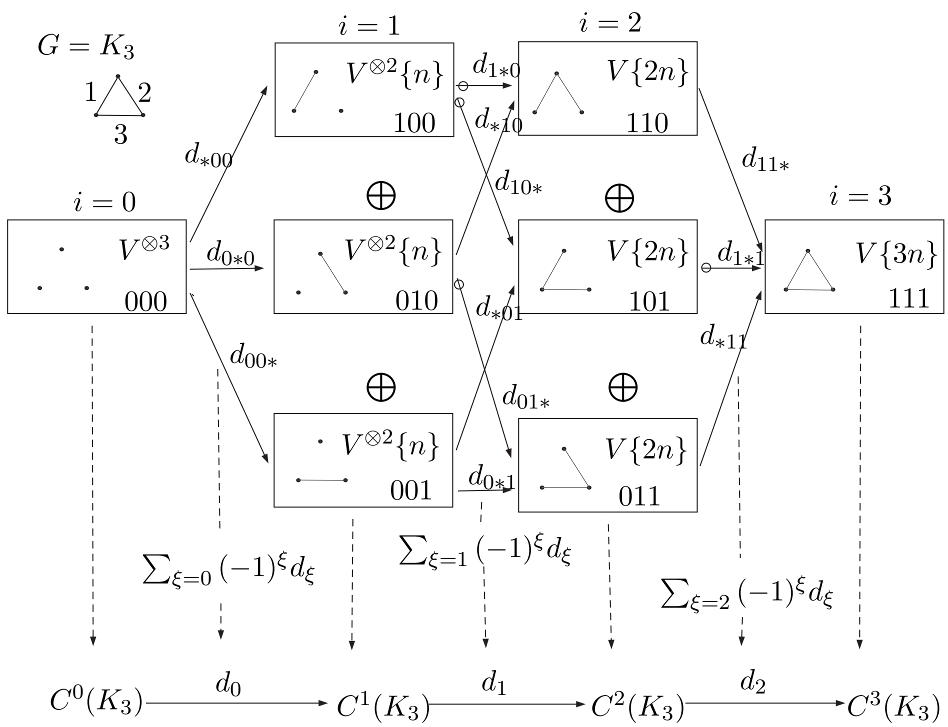


Figure 3.2: The chain complex and the differentials

To get the chain groups we "flatten" the cube by taking direct sums along the columns. More precisely:

Definition 6 *We set the i-th chain group $C_n^i(G)$ of the chain complex $\mathcal{C}_n(G)$*

to be the direct sum of all $\mathbb{Z}$-modules at height i, i.e. $C_n^i(G) = \oplus_{|\epsilon|=i} V_\epsilon(G)$. The grading is given by the degree of the elements and we can write the i-th chain group as $C_n^i(G) = \sum_{j\geq 0} C_n^{i,j}(G)$, where $C_n^{i,j}(G)$ denotes the set of elements of degree j of $C_n^i(G)$.

So, we are left with defining the grading preserving differential, since then according to Proposition 2 and Remark 1 (see subsection 2.4.2), the chain complex obtained will have the n-specialization of the dichromatic polynomial as its Euler characteristic.

3.3.1 The differential

We define the differential in the, now, standard way. We are first going to define per-edge maps between some vertices of the cube $\{0,1\}^m$ – the maps that correspond to the edges of the cube. We define them as linear, degree preserving maps and such that the cube is commutative, i.e. every square is commutative. Then we build the differential by summing with appropriate signs along columns, and hence obtain a map whose square is zero. This is presented graphically in Figure 3.2.

So, let us first define per-edge maps. Each vertex of the cube $\{0,1\}^m$ is labeled with some $\epsilon = (\epsilon_1, \ldots, \epsilon_m) \in \{0,1\}^m$. There are maps between two vertices only if one of the markers ϵ_i is changed from 0 to 1 when one goes from the first vertex to the second vertex and all the other ϵ_i are unchanged.

Denote by ϵ the label of the first vertex. If the marker which is changed from 0 to 1 has index j then the map will be denoted by $d_{\epsilon'}$, where $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_m)$ with $\epsilon'_i = \epsilon_i$ if $i \neq j$ and $\epsilon'_i = *$ if $i = j$. Denote by $|\epsilon|$ the sum of components of ϵ, and by l the number of connected components of G_ϵ. Changing exactly one marker from 0 to 1 corresponds to adding an edge. That edge can either connect two different connected components, or it can connect one component to itself.

In the first case, we have that two components are merging into one and the remaining $l-2$ components are left the same. We have to define a degree preserving map from $V^{\otimes l}\{n|\epsilon|\}$ to $V^{\otimes(l-1)}\{n(|\epsilon|+1)\}$. We define it to be the identity on the $l-2$ tensor factors corresponding to the connected components which are left unchanged and on the remaining tensor product of two copies of V we have to define a map from $V \otimes V$ to $V\{n\}$. We define it on the basis vectors as:

$$X^i \otimes X^j \mapsto X^{i+j}, \quad i,j = 0, \ldots, n,$$

where we assume $X^i = 0$, for $i > n$.

In the second case, the added edge belongs to one connected component,

say the p-th one, and the remaining components will remain unchanged. In this case, we have to define a degree preserving map from $V^{\otimes l}\{n|\epsilon|\}$ to $V^{\otimes l}\{n(|\epsilon|+1)\}$. We define the map as the identity on those $l-1$ copies of V which correspond to the connected components which are not affected by adding the new edge, while on the remaining copy of V (the one which corresponds to the p-th connected component), we have to define a linear map from V to $V\{n\}$. We define it on the basis vectors by $X^i \mapsto 0$, for $i = 1, \ldots, n$, and $1 \mapsto X^n$.

Remark 5 *We could also define the differential in the second case as the zero map (i.e. $1 \mapsto 0$ as well). This would also define a degree preserving map, and the theory would not be trivial since the differential in the first case (the added edge connects two different components) is nontrivial.*

The per-edge differentials obtained in this way obviously make the cube commutative. We will sprinkle signs around on the edges of the cube and thus obtain a map whose square is zero. Namely, we define the differential $d^i : C_n^i(G) \to C_n^{i+1}(G)$ on the chain complex $\mathcal{C}_n(G)$ by

$$d^i := \sum_{|\epsilon|=i} (-1)^{\epsilon} d_{\epsilon},$$

where we sum over all m-tuples of 0's, 1's, and exactly one $*$. Here we denoted the number of 1's in ϵ by $|\epsilon|$, and $(-1)^{\epsilon}$ equals -1 if there is odd number of 1's before $*$ in ϵ, and $+1$ if there is even number of them. In Figure 3.2 we indicated the differentials with minus sign by adding a small circle at the tail of the arrow.

A straightforward calculation implies:

Proposition 3 *This defines a differential, that is, $d^{i+1}d^i = 0$.*

Now we really have a chain complex $\mathcal{C}_n(G)$ where the chain groups and the differential are defined as in the previous two paragraphs, and according to Proposition 2, we have

Theorem 6 *The Euler characteristic of the chain complex $\mathcal{C}_n(G)$ is equal to the n-specialization $P_{G,n}$ of the dichromatic polynomial of the graph G.*

Even though our construction depends on the ordering of the edges of the graph, in exactly the same way as in [15], section 2.2.3, we obtain the following:

Theorem 7 *The homology groups of the chain complex $\mathcal{C}_n(G)$ are graph invariants.*

For each graph G and positive integer n, we define the (two-variable) Poincaré polynomial of the corresponding chain complex $\mathcal{C}_n(G)$, in the standard way, by $R_{G,n}(q,t) = \sum_{i\in\mathbb{Z}} t^i q \dim(H^i)$. Then from Theorems 6 and 7 we obtain:

Proposition 4 *a) For each positive integer n, the polynomial $R_{G,n}$ is a graph invariant.*
b) The n-specialization of the dichromatic polynomial $P_{G,n}(q)$ is equal to $R_{G,n}(q,-1)$.

3.3.2 Some calculations

In this subsection we give the values of our homology theory for the polygon graph $\mathcal{P}_k$ with k vertices and k edges. The results from the following theorem are for the homology theory where the differentials in the case when the added edge connects one connected component to itself is the zero map (see Remark 5). In the general case, one can easily obtain an analogous result.

Theorem 8 Odd case: *If $k = 2g+1$, the free part of the homology is given by the Poincaré polynomial:*

$$\begin{gathered}(1+q+\ldots+q^{n-1})^{2g+1} + \\ (1+q+\ldots+q^{n-1})(\textstyle\sum_{i=1}^{g-1}(t^{2i-1}+t^{2i})q^{gn-g+i(n+1)} + t^{2g-1}q^{2gn}) + \\ t^{2g}q^{2gn}(1+q+\ldots+q^n) + t^{2g+1}q^{(2g+1)n}(1+q+\ldots+q^n).\end{gathered}$$

The torsion part is given by:

$$Tor(H_n^{*,*}(\mathcal{P}_{2g+1})) = \bigoplus_{i=1}^{g} H_n^{2i-1,gn-g-1+i(n+1)}(\mathcal{P}_{2g+1}),$$

with each summand being isomorphic to $\mathbb{Z}_{n+1}$.
Even case: *If $k = 2g+2$, the free part of the homology is given by the Poincaré polynomial:*

$$\begin{gathered}(1+q+\ldots+q^{n-1})^{2g+2} + \\ (1+q+\ldots+q^{n-1})(\textstyle\sum_{i=0}^{g-1}(t^{2i}+t^{2i+1})q^{gn+n-g+i(n+1)} + t^{2g}q^{(2g+1)n}) + \\ t^{2g+1}q^{(2g+1)n}(1+q+\ldots+q^n) + t^{2g+2}q^{(2g+2)n}(1+q+\ldots+q^n).\end{gathered}$$

The torsion part is given by:

$$Tor(H_n^{*,*}(\mathcal{P}_{2g+2})) = \bigoplus_{i=1}^{g} H_n^{2i,(g+1)(n-1)+i(n+1)}(\mathcal{P}_{2g+2}),$$

with each summand being isomorphic to $\mathbb{Z}_{n+1}$.

Proof:

First of all, note that up to the homological degree $k-2$, all the added edges connect two different components, and hence their contribution to the differential is only of that type (algebra multiplication). Also, that part of the construction coincides with the construction [16] and [17] for the algebra $\mathcal{A}_{n+1}$, and the only difference is in the grading. It is easy to see that the element of q-degree i in [16] becomes the element of q-degree $(2g+1)n-i$ in our theory, and so by applying the result from [40], Corollary 4.3, we obtain the homology groups $H^i(\mathcal{P}_k)$, for $i = 0, \ldots, k-2$.

Since we took the case when the second type of the differentials are zero maps, we have that $H^k(\mathcal{P}_k) = V\{kn\}$. Finally, the remaining cohomology group $H^{k-1}(P_k)$ we determine by using the fact that the restriction $t = -1$ (graded Euler characteristic) is equal to the n-specialization $P_{G,n}$ of the dichromatic polynomial. ■

Remark 9 *We could also obtain the result of the previous theorem by using the Hochschild homology approach directly, like in [40].*

3.4 An "infinite-dimensional" set of specializations

The set of specializations of the dichromatic polynomial that we have defined in the previous section is enough to recover the original, two-variable dichromatic polynomial, but it "misses" the specialization that corresponds to the Jones polynomial. In this section we define an alternative set of one-variable specializations (one polynomial for every integer $n \le 2$) and the corresponding set of chain complexes that categorify them. Although the categorification of this set of specializations has chain groups of infinite rank, the advantage is that the case $n = 2$ is equal (up to a multiple) to the Jones polynomial of an alternating link that corresponds to the planar graph.

For every integer $n \le 2$, we define the polynomial $Q_{G,n}$ by

$$Q_{G,n}(q) = P_G(q, q^n/(q-1)).$$

Note that in this case we have $v = q^n/(q-1) = q^{n-1}\sum_{i\geq 0} q^{-i} = q_n$ in axiom $(A2)$.

The categorification of Q_n follows the same lines as the categorification in Section 3.3. Namely, we organize the summands of the state-sum expression for Q_n in the same cube as in figure 3.1, with the replacement of $\{n\}_q$ by q_n and of q^n by q. Furthermore, to each $\epsilon \in \{0,1\}^m$ (i.e. to each vertex of the cube of resolutions) we assign a $\mathbb{Z}$-module M_ϵ in the following way: First of all, we order the vertices of G, and to the i-th one, we assign the index i. Let K_1, K_2,..., K_l be the set of connected components of G_ϵ. Then to the component K_j we will assign the variable x_{i_j}, where i_j is the smallest index among the vertices v that belong to K_j. Finally, we define $M_\epsilon(G) = \mathbb{Z}[x_{i_1}, \dots, x_{i_l}]\{l(n-1) + |\epsilon|\}$.

In other words, to each connected component of G_ϵ we assign a copy of $M = \mathbb{Z}[x]\{n-1\}$, then take the tensor product and finally increase the degree by $|\epsilon|$. From the definition we have that $q\dim M_\epsilon(G)$ is (up to the sign $(-1)^{|\epsilon|}$) the contribution of Q_{G_ϵ} to the dichromatic polynomial (see Example 2 in Section 3.3).

In terms of pictures, we make the same picture as in figure 3.2, only replacing V by M and all shifts $\{ni\}$ by $\{i\}$. Finally the chain groups $D^i_n(G)$ are obtained by summing along the columns, i.e. $D^i_n(G) = \oplus_{|\epsilon|=i} M_\epsilon(G)$.

Now, we move to the differential. The prescription is the same as in the previous section. We only have to define per-edge maps and make the cube commutative, and then the signs (defined in the same way as previously) will make the cube anti-commutative, which after summing over columns defines the map whose square is equal to zero, i.e. the differential.

Again, the edges of the cube of resolutions correspond to adding an edge of the graph G. This can affect the connected components of the resolution in two ways: either the added edge connects two components (and consequently decreases the number of components by 1) or the added edge belongs to one of the components (and hence preserves the number of components).

If adding the edge decreases the number of components by one, then we have that two components (say the p-th and the q-th one) merge into one, and the remaining $l-2$ components are left the same. Suppose that $i_p < i_q$. We have to define a degree preserving map from $M_\epsilon = \mathbb{Z}[x_{i_1}, \dots, x_{i_l}]\{l(n-1)\}$ to $M_{\epsilon'} = \mathbb{Z}[x_{i_1}, \dots, \hat{x}_{i_q}, \dots, x_{i_l}]\{(l-1)(n-1)+1\}$, where by $\hat{x}_{i_q}$ we have indicated that x_{i_q} is omitted. We define it on the basis monomials by:

$$\prod_{j=1,\dots,l} x_{i_j}^{a_j} \mapsto \prod_{\substack{j=1,\dots,l \\ j\neq q}} x_{i_j}^{b_j},$$

where $b_i = a_i$, for $i \neq p$, and $b_p = a_p + a_q + 2 - n$. In other words, we map x_{i_q} to x_{i_p} and multiply the result by $x_{i_p}^{2-n}$.

If adding that edge does not affect the number of components, then it means that the added edge will belong to some component, say the p-th one, and that the remaining components will remain unchanged. In this case, we have to define a grading preserving map from $M_\epsilon = \mathbb{Z}[x_{i_1}, \ldots, x_{i_l}]\{l(n-1)\}$ to $M_{\epsilon'} = \mathbb{Z}[x_{i_1}, \ldots, x_{i_l}]\{l(n-1)+1\}$. We define this map as the multiplication by x_{i_p}.

Remark 10 *Like in the case of the first set of specializations from the previous Section, we could define the differential in the second case (number of connected components is preserved) as the zero map. Since in the first case the differential is nontrivial, this again defines a chain complex with nontrivial degree-preserving differentials.*

These per-edge maps obviously make the cube commutative and after defining signs in completely the same way as before, and adding over the columns (as in figure 3.2), we obtain the differential d of our chain complex $D_n(G)$. Again, as before we have that the homology groups $H^i(D_n(G))$ do not depend on the ordering of edges of the graph G, and since the differential is degree preserving, we have:

Theorem 11 *The homology groups of the chain complex $D_n(G)$ are graph invariants. The graded Euler characteristic of the complex $D_n(G)$ is equal to the n-th specialization $Q_n(G)$ of the dichromatic polynomial of the graph G.*

Remark 12 *We could also define per-edge differentials in the finitedimensional case in Section 3.3 in the same manner as above, by using the alternative description of V as a quotient of the ring of polynomials. Namely if the added edge connects two components, then the differential is given on the corresponding tensor product of two copies of V by multiplication of polynomials (and setting X_q to be X_p), and in the other case is given by multiplication by X_p^n (or as the zero map).*

Remark 13 *Since both our constructions of categorification follow the same lines of the cubic complex construction, as it is used in other graph or link homologies ([5], [24], [30], [15], [16], [17], [57], [34]), we also obtain the long exact sequence of homology groups:*

$$\cdots \to H^{i-1}(G/e) \to H^i(G) \to H^i(G-e) \to H^i(G/e) \to H^{i+1}(G) \to \cdots$$

The analogous long exact sequence was first explicitly observed in [57], but it is also implicit in [24], since chain complexes in all link (and graph) homology constructions are in fact mapping cones of certain homomorphisms, and hence we always obtain the long exact sequence in homology.

3.5 Categorification of the Jones polynomial for alternating links

As is well-known, every planar graph is in bijective correspondence with some knot universe (knot shadow), which in turn gives rise to an alternating link diagram, up to mirror image. As it is also known (e.g. [8]), the Jones polynomial of an alternating link is related (equal up to a multiple) with the value of the Tutte polynomial $T_G(x, 1/x)$ of a planar graph corresponding to it. As we saw in the introduction the value of this specialization of the Tutte polynomial is (up to a multiple) equal to $P_G(q, q^2/(q-1))$ (with $q = 1 - 1/x$). This can also be seen directly (see Appendix A and e.g. [22]). Hence, we can obtain a categorification of the Jones polynomial of an alternating link, by categorifying the one-variable specialization $J_G(q) = P_G(q, q^2/(q-1))$ of the dichromatic polynomial. As we saw in the previous Section, we managed to categorify an infinite set of specializations of the dichromatic polynomial (enough to recover it), and for $n = 2$ we obtain that the specialization coincides with $J_G(q)$. However, in this section we will give an explicit categorification for this graph (Laurent) polynomial in terms of enhanced states.

First of all, notice that we can write the value v (which in the case of $J_G(q)$ is equal to $q^2/(q-1)$) as the following series:

$$v = q/(1-q^{-1}) = q\sum_{i\geq 0} q^{-i} = \sum_{i\geq 0} q^{1-i}, \tag{3.2}$$

and with that value of v, the polynomial $J_G(q)$ is given by

$$J_G(q) = \sum_{s\subset E(G)} (-1)^{|s|} q^{|s|} v^{k(s)}.$$

In order to define the graded complex, we will introduce the notion of enhanced states of a graph G. An enhanced state S is given by a pair $S = (s, l)$, where s is a state (subset of the set of edges $E(G)$), and l is a labeling of the connected components of $[G : s]$ by nonnegative integers, i.e. a sequence of $k(s)$ nonnegative integers $l_1, \ldots, l_{k(s)}$. Denote by $|l|$, the

sum of all l_i's, $i = 1, \ldots, k(s)$. To each enhanced state S we will assign two numbers: $i(S) = |s|$, and $j(S) = |s| + k(s) - |l|$. Now, we define $C^{i,j}(G)$ (the j-th graded component of the i-th chain group, $0 \le i \le m$, $j \in \mathbb{Z}$) as the span over $\mathbb{Z}$ of all enhanced states S of G such that $i(S) = i$ and $j(S) = j$.

We will define a (degree preserving) differential $d : C^{i,j}(G) \to C^{i+1,j}(G)$, by giving its value on each enhanced state $S = (s, l) \in C^{i,j}(G)$. We define:

$$d(S) = \sum_{e \in E(G) \setminus s} (-1)^{n_e} S_e, \tag{3.3}$$

with n_e being the number of edges in s ordered before e, and where$S_e = (s_e, l_e)$ is an enhanced state whose state is $s_e = s \cup \{e\}$ and whose labeling l_e is defined as follows: denote by $E_1, \ldots, E_k$ the connected components of $[G : s]$.

If e connects some E_i to itself then the components of $[G : s_e]$ are $E_1, \ldots, E_i \cup \{e\}, \ldots, E_k$, and we define the labeling l_e by $l_e(E_i \cup \{e\}) = l(E_i) + 1$ and $l_e(E_j) = l(E_j)$, for $j \neq i$.

If e connects some E_i to E_j (say E_1 and E_2), then the components of $[G : s_e]$ are $E_1 \cup E_2 \cup \{e\}, E_3, \ldots, E_k$, and we define the labeling l_e by $l_e(E_1 \cup E_2 \cup \{e\}) = l(E_1) + l(E_2)$ and $l_e(E_j) = l(E_j)$, for $j > 2$.

Proposition 5 *The mapping d, defined by (3.3), satisfies $d^2 = 0$.*

Proof:

Let i and j be arbitrary integers, and let S be an enhanced state from $C^{i,j}$. If e and f are two distinct edges of $E(G) \setminus s$, then from the definition of the states S_e in the image of the differential, we have $(S_e)_f = (S_f)_e$. Suppose that we have introduced an ordering $<$ on the set $E(G)$. Also, denote the number of edges in $s \cup \{f\}$ ordered before e by n'_e, and the number of edges in $s \cup \{e\}$ ordered before f by n'_f. Now we have:

$$\begin{aligned} d(d(S)) &= d(\sum_{e \in E(G) \setminus s} (-1)^{n_e} S_e) = \\ &= \sum_{e \in E(G) \setminus s} \left(\sum_{f \in E(G) \setminus s_e} (-1)^{n'_f} (-1)^{n_e} (S_e)_f \right) = \\ &= \sum_{\substack{e, f \in E(G) \setminus s \\ e < f}} ((-1)^{n'_f} (-1)^{n_e} + (-1)^{n'_e} (-1)^{n_f}) (S_e)_f = 0, \end{aligned}$$

as required. ■

In this way we have obtained a sequence of chain complexes (one for each degree j). We could take the direct sum along columns (fixed i), and obtain a (simply) graded chain complex $\mathcal{C}(G)$ with chain groups given by:

$$C^i(G) = \oplus_{j\in\mathbb{Z}} C^{i,j}(G).$$

The graded dimension of $C^i(G)$ is equal to the sum of the graded dimensions corresponding to each state s with $|s| = i$. From the definition of the gradings of our enhanced states we have that the graded dimension corresponding to each such state s is equal to

$$q^i(\sum_{j\geq 0} q^{1-j})^{k(s)} = q^i v^{k(s)},$$

with v given by (3.2). Furthermore, since we have defined degree preserving differentials, we have the following:

Theorem 14 *The graded Euler characteristic of the chain complex $\mathcal{C}(G)$ is equal to $J_G(q)$.*

Chapter 4

Koszul complexes and categorification of link and graph invariants

4.1 Triply graded link homology

4.1.1 Introduction

In this section we will introduce the parametrization of the HOMFLYPT polynomial that we will categorify. It is very similar to the one in [31]. Throughout the chapter we will consider only braid diagrams D of a link L, i.e. regular diagrams which are the closures of upward oriented braids (see Section 2.2).

As is well known, every link can be represented by a braid diagram. Also, the closures of two braid diagrams D_1 and D_2 are isotopic as oriented links if and only if D_1 and D_2 are related by a sequence of Markov moves, which are the following (see [37]):

(i) conjugation: $DD' \longleftrightarrow D'D$

(ii) transformations in the braid group:

$$\begin{array}{rcl} D & \longleftrightarrow & D\sigma_i\sigma_i^{-1} \\ D & \longleftrightarrow & D\sigma_i^{-1}\sigma_i \\ D\sigma_j\sigma_i & \longleftrightarrow & D\sigma_i\sigma_j, \quad |i-j|>1 \\ D\sigma_i\sigma_{i+1}\sigma_i & \longleftrightarrow & D\sigma_{i+1}\sigma_i\sigma_{i+1} \end{array}$$

(iii) transformations $D \longleftrightarrow D\sigma_n^{\pm 1}$, for a braid D with n strands.

In order to define the HOMFLYPT polynomial for a link L, from its braid diagram representation D, we will introduce a function F on braid diagrams with values in the ring of rational functions in q and t defined uniquely by the following axioms:

$*$ $F(D_1) = F(D_2)$, if D_1, D_2 are related by Markov move (i)
$*$ $F(D_1) = F(D_2)$, if D_1, D_2 are related by Markov moves (ii)
$*$ $F(D\sigma_n) = F(D)$, if a braid D has n strands
$*$ $F(D\sigma_n^{-1}) = -t^{-1}q^{-1}F(D)$, if a braid D has n strands
$*$ Skein relation: for every braid diagram D with n strands and $0 < i < n$

$$q^{-1}F(D\sigma_i) - qF(D\sigma_i^{-1}) = (q^{-1} - q)F(D)$$

$*$ If U is the one-strand diagram of the unknot then $F(U) = 1$.

In order to obtain a link invariant we need to normalize the function F. Let $\alpha = -t^{-1}q^{-1}$ and let

$$G(D) = \sqrt{\alpha}^{n_+(D)-n_-(D)-s(D)+1}F(D), \tag{4.1}$$

where $n_+(D)$, $n_-(D)$ and $s(D)$ are the number of positive crossings, negative crossings and the number of strands of D, respectively. We denote the expression $n_+(D)-n_-(D)-s(D)+1$ by $\omega(D)$. Obviously $G(D)$ is invariant under all Markov moves of braids and it satisfies the HOMFLYPT skein relation

$$(q\sqrt{\alpha})^{-1}G(D\sigma_i) - q\sqrt{\alpha}G(D{\sigma_i}^{-1}) = (q^{-1} - q)G(D).$$

Hence, $G(D)$ is equal to the HOMFLYPT polynomial of the link L, normalized such that $G(U) = 1$. In Section 4.1.3, we will define a triply graded chain complex $\mathcal{C}(D)$ whose Euler characteristic is equal to $F(D)$.

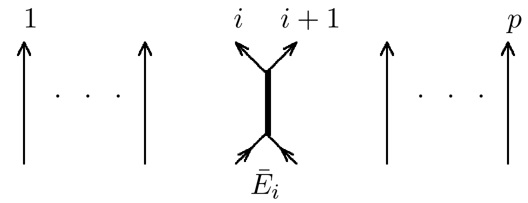
First of all note our slightly different convention compared to [31] on the value of unknot. This has the advantage that we can obtain the Alexander polynomial directly by specializing t and q ($t = -q$), and the whole sequence of the n-specializations of the (reduced) HOMFLYPT polynomial (see [30], [38]). Specifically, by taking $t = -q^{1-2n}$ we obtain polynomials $G_n(D)$ that satisfy the skein relation

$$q^{-n}G_n(D\sigma_i) - q^nG_n(D{\sigma_i}^{-1}) = (q^{-1} - q)G_n(D),$$

and whose value on the unknot is $G_n(U) = 1$. Hence by suitably collapsing the tri-grading to a bi-grading we get a new categorification of the n-specializations of the HOMFLYPT polynomial.

4.1.2 Graphs with wide edges

In order to define the function $F(D)$ and hence the HOMFLYPT polynomial $G(D)$, we introduce the trivalent graphs with wide edges (like in Chapter 2, Section 2.5) as the resolutions of the crossings. Apart from the crossings σ_i and σ_i^{-1}, we introduce wide edges $\bar{E}_i$ placed between the i-th and $(i+1)$-th strand of the braid, like in the following picture:



Then we can define the function $F(D)$ by resolving the crossings by using the following two relations:

$$F(D\sigma_i) = F(D\bar{E}_i) - q^2 F(D), \tag{4.2}$$

$$F(D{\sigma_i}^{-1}) = q^{-2} F(D\bar{E}_i) - q^{-2} F(D). \tag{4.3}$$

Here by $F(D)$ we mean the value of the function F on the diagram that is the closure of the braid diagram D, and we have extended the domain of the function F to include trivalent graphs. Then F (restricted to braid diagrams) will satisfy the axioms from the previous subsection, if and only if the values of F on completely resolved trivalent graphs satisfy

$$F(U) = 1 \tag{4.4}$$

$$F(D \cup U) = \tfrac{1+t^{-1}q}{1-q^2} F(D), \quad \text{if } D \text{ is not an empty diagram} \tag{4.5}$$

$$F(D\bar{E}_n) = \tfrac{1+t^{-1}q^3}{1-q^2} F(D), \quad \text{where } D \text{ is a diagram with } n \text{ strands} \tag{4.6}$$

$$F(D\bar{E}_i^2) = (1+q^2)F(D\bar{E}_i) \tag{4.7}$$

$$F(D\bar{E}_i\bar{E}_{i+1}\bar{E}_i) + q^2 F(D\bar{E}_{i+1}) = F(D\bar{E}_{i+1}\bar{E}_i\bar{E}_{i+1}) + q^2 F(D\bar{E}_i). \tag{4.8}$$

We will use (4.2) and (4.3) in resolving the crossings in the definition of the triply-graded chain complex that categorifies the HOMFLYPT polynomial. For some more details about state-sum models of the HOMFLYPT polynomial, see Appendix B.

4.1.3 Categorification of the two-variable HOMFLYPT polynomial

In this subsection we will give an alternative, simpler and (essentially) equivalent construction to the one in [31] (similar simplification is also implicit

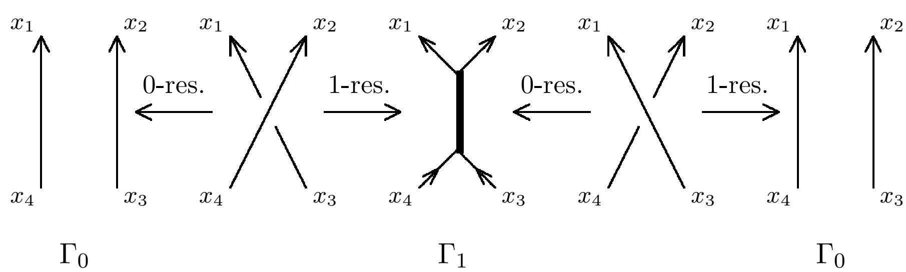
in [29]).

Essentially, we will set the variable a from [31] to be 0, but we will keep the double grading of the ring of polynomials $R' = \mathbb{Q}[x_1, \dots, x_{2n}]$. Specifically, to every arc (line between two crossings) we will assign a different variable x_i, $i = 1, \dots, 2n$, where n is the number of crossings of a given planar projection D of a knot K. We define the bidegree of every x_i to be (0,2) and we put the field of coefficients $\mathbb{Q}$ in bidegree (0,0). Also, by $\{\cdot, \cdot\}$ we denote a shift in bigrading (like in the case of a simply graded space, see Section 2.4).

Remark 15 *Note that this corresponds to the case* $n = -1$ *in [30], but with the introduction of a new grading direction.*

Let L be a link and let D be its braid diagram presentation. Let I be the ideal of R' generated by the monomials $x_1 + x_2 - x_3 - x_4$ for every crossing of D, and let $R = R'/I$ (see the picture below for the notation).

To each crossing we will assign 0- and 1-resolutions according to the following picture (the same as in Section 2.6):



and we call the resolutions obtained Γ_0 and Γ_1, as written in the picture. To the resolution Γ_0 we assign the following complex:

$$\mathcal{C}(\Gamma_0): \qquad 0 \longrightarrow R\{-1,1\} \xrightarrow{x_2 - x_3} R \longrightarrow 0$$

and to the resolution Γ_1 with the wide edge we assign the complex:

$$\mathcal{C}(\Gamma_1): \qquad 0 \longrightarrow R\{-1,3\} \xrightarrow{(x_2-x_3)(x_4-x_2)} R \longrightarrow 0.$$

Assume that there are no free circles in the diagram D. If we resolve all the crossings of D we obtain a trivalent graph with wide edges. There are 2^n such resolutions Γ of D and to each one we assign the tensor product of $\mathcal{C}(\Gamma_0)$ and $\mathcal{C}(\Gamma_1)$, over all crossings $c(D)$, depending on the type of resolution that appeared. In this way we have obtained a complex $\mathcal{C}(\Gamma)$ and to each resolution Γ we will assign its cohomology $H(\Gamma) = H(\mathcal{C}(\Gamma))$.

Like in [31], we can obtain that $H(\Gamma)$ categorifies the relations (4.5)–(4.8). For example, the relation (4.5) becomes

$$H(\Gamma \cup \text{ unknot}) \cong (H(\Gamma) \otimes \mathbb{Q}[x_i]) \oplus (H(\Gamma) \otimes \mathbb{Q}[x_i]\{-1,1\}),$$

where x_i is the label assigned to the circle (unknot). Note that in all definitions only the differences $x_i - x_j$ appear. Thus we can work with the smaller ring of polynomials $R'' = \mathbb{Q}[x_2 - x_1, \ldots, x_{2n} - x_1]$ instead of R' (like in [31]).

If we have free circles in the the diagram D, we introduce new variable y, with $\deg y = (0,2)$, extend the ring of polynomials to $R[y]$ and replace R by $R[y]$ in the complexes $\mathcal{C}(\Gamma_i)$, $i = 0,1$. Finally to every free circle we assign the complex:

$$0 \longrightarrow R'[y]\{-1,1\} \xrightarrow{y} R'[y] \longrightarrow 0,$$

and we tensor these complexes with $\mathcal{C}(\Gamma)$. In this way we obtain good value of the unknot (4.4), i.e. $H(U) \cong \mathbb{Q}$.

We again organize the 2^n total resolutions Γ of the diagram D in the same cubic complex as in the standard categorifications. To each vertex of the cube (i.e. to each total resolution Γ) we assign the graded vector space $H(\Gamma)$.

We will introduce the differentials between those cohomology groups as the maps induced by the (grading preserving) homomorphisms between the corresponding complexes $\mathcal{C}(\Gamma)$. Since these complexes are built as the tensor products of $\mathcal{C}(\Gamma_0)$ and $\mathcal{C}(\Gamma_1)$ it is enough to specify the homomorphisms between these two complexes. For a positive crossing c we define the following complex of complexes:

$$\mathcal{C}_c : \qquad 0 \longrightarrow \mathcal{C}(\Gamma_0)\{0,2\} \xrightarrow{\chi_0} \mathcal{C}(\Gamma_1) \longrightarrow 0, \tag{4.9}$$

where $\mathcal{C}(\Gamma_1)$ is in cohomological degree 0, and the map χ_0 is given by

$$\begin{array}{ccccccc}
0 & \longrightarrow & R\{-1,3\} & \xrightarrow{x_2-x_3} & R\{0,2\} & \longrightarrow & 0 \\
 & & {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle x_4-x_2} & & \\
0 & \longrightarrow & R\{-1,3\} & \xrightarrow{(x_2-x_3)(x_4-x_2)} & R & \longrightarrow & 0.
\end{array}$$

For a negative crossing c we define the following complex of complexes:

$$\mathcal{C}_c : \qquad 0 \longrightarrow \mathcal{C}(\Gamma_1)\{0,-2\} \xrightarrow{\chi_1} \mathcal{C}(\Gamma_0)\{0,-2\} \longrightarrow 0, \tag{4.10}$$

where $\mathcal{C}(\Gamma_1)$ is in cohomological degree 0, and the map χ_1 is given by

$$\begin{array}{ccccccc}
0 & \longrightarrow & R\{-1,1\} & \xrightarrow{(x_2-x_3)(x_4-x_2)} & R\{0,-2\} & \longrightarrow & 0 \\
 & & {\scriptstyle x_4-x_2}\big\downarrow & & \big\downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & R\{-1,-1\} & \xrightarrow{x_2-x_3} & R\{0,-2\} & \longrightarrow & 0.
\end{array}$$

Define $\mathcal{C}(D)$ as the tensor product of $\mathcal{C}_c$ over all crossings $c(D)$. It is a complex built out of Koszul complexes $\mathcal{C}(\Gamma)$, over all the total resolutions Γ of the diagram D, and the differential preserves the bigrading of each term $\mathcal{C}^j(D)$. Every $\mathcal{C}^j(D)$ decomposes as a direct sum of contractible two-term complexes and its cohomology $H(\mathcal{C}^j(D))$, which is denoted by $\mathcal{C}H^j(D)$. The differential induces the grading preserving maps δ from $\mathcal{C}H^j(D)$ to $\mathcal{C}H^{j+1}(D)$ and we denote the complex obtained in this way by $\mathcal{C}H(D)$. The cohomology $H(D) = H(\mathcal{C}H(D), \delta)$ is triply-graded:

$$H(D) = \bigoplus_{j,k,l} H^j_{k,l}(D).$$

Here j is the cohomology degree, and k and l come from the internal bigrading of the chain groups.

In complete analogy with [31] we have that $H(D)$ does not depend on the choice of the braid presentation D of a link L, up to an overall shift in the triple grading. Also, as we saw since $H(\Gamma)$ categorifies the relations (4.4)–(4.8) and since the differentials are induced by the grading preserving maps (4.9) and (4.10) which obviously categorify the relations (4.2) and (4.3), we have that the bigraded Euler characteristic of $\mathcal{C}H(D)$ is equal to $F(D)$. Finally, by introducing half-integral shifts as in [59] (in order to compensate the powers of α from (4.1)) by:

$$\mathcal{H}(D) = H(D)[\omega(D)/2]\{-\omega(D)/2, -\omega(D)/2\},$$

we obtain a triply-graded homology theory, which does not depend on the choice of the braid presentation D of a link L and whose bi-graded Euler characteristic is equal to the two-variable HOMFLYPT polynomial of a link L.

4.2 New categorifications of the chromatic and dichromatic polynomials for graphs

4.2.1 Introduction

In Chapter 3 we categorified the dichromatic polynomial of graphs by introducing one-variable specializations of it and then we categorified each of these one-variable polynomials. In this section we will define a complex of doubly-graded modules whose doubly-graded Euler characteristic is equal to the whole two-variable dicromatic polynomial. The idea is partially inspired by the categorification of HOMFLYPT polynomial described in the previous section.

Also, we give a new categorification of the chromatic polynomial for graphs. We do this here differently to [15]. We will define the chain groups (the modules corresponding to the vertices of the cube of resolutions) as the cohomologies of certain chain complexes.

We will use the same notation as in Chapter 3: namely, a graph G is specified by a set of vertices $V(G)$ and a set of edges $E(G)$. If e is an arbitrary edge of the graph G, then by $G - e$ we denote the graph G with the edge e deleted, and by G/e the graph obtained by contracting edge e (i.e. by identifying the vertices incident to e and deleting e).

4.2.2 The chromatic polynomial

If q is a positive integer, the chromatic polynomial $P_G(q)$ is defined as the number of ways to color the vertices of G by using at most q colors, such that every two vertices which are connected by an edge receive a different color. It is well-known that the chromatic polynomial can be defined equivalently by the following two axioms:

$$\begin{aligned} &(C1) \qquad P_G = P_{G-e} - P_{G/e}, \\ &(C2) \qquad\qquad P_{N_k} = q^k, \end{aligned}$$

where N_k is the graph with k vertices and no edges. By using these axioms we can obviously extend the domain of the polynomial to the set of complex numbers, and, furthermore, instead of q in the axiom $(C2)$ we will put $1/(1-q)$, with $|q| < 1$.

By repeated use of $(C1)$ (which is the famous deletion-contraction rule) we will obtain the value of the chromatic polynomial as a sum of contributions from all spanning subgraphs of G (subgraphs that contain all vertices of G), which we will call states. Furthermore, if for each subset $s \subset E(G)$

we denote by $[G : s]$ the graph whose set of vertices is $V(G)$ and set of edges is s, then the contribution of the graph $[G : s]$ is $(-1)^{|s|}(1-q)^{-k(s)}$, where $|s|$ is the number of elements of s and $k(s)$ is the number of connected components of $[G : s]$. Hence, we obtain the expression:

$$P_G(q) = \sum_{s \subset E(G)} (-1)^{|s|}(1-q)^{-k(s)} = \sum_{i \geq 0} (-1)^i \sum_{s \subset E(G), |s|=i} (1-q)^{-k(s)},$$

which is called the state-sum expansion of the polynomial $P_G(q)$.

In subsection 4.2.4 we will define a graded chain complex of modules $\mathcal{C}(G)$ whose graded Euler characteristic is equal to $P_G(q)$.

4.2.3 The dichromatic polynomial

The dichromatic polynomial $P_G(q, v)$ of the graph G is a two-variable generalization of the chromatic polynomial given by the following two axioms:

$$\begin{array}{ll} (D1) & P_G = P_{G-e} - qP_{G/e}, \\ (D2) & P_{N_k} = v^k, \end{array}$$

where N_k is the graph with k vertices and no edges.

From $(D1)$ we have a recursive expression for the dichromatic polynomial in terms of the value of the polynomial on graphs with a smaller number of edges. Indeed, as in the case of the chromatic polynomial we obtain that the contribution of the state $[G : s]$ is $(-1)^{|s|}q^{|s|}v^{k(s)}$, where $|s|$ is the number of elements of s and $k(s)$ is the number of connected components of $[G : s]$. Hence, we obtain the expression:

$$P_G(q, v) = \sum_{s \subset E(G)} (-1)^{|s|}q^{|s|}v^{k(s)} = \sum_{i \geq 0} (-1)^i q^i \sum_{s \subset E(G), |s|=i} v^{k(s)},$$

which is called the state-sum expansion of the polynomial $P_G(q, v)$. However, we will use a slightly different parametrization of the dichromatic polynomial, given by:

$$D_G(t, q) = (1 + t^{-1}q)^m P_G(q, \frac{1 + t^{-1}q}{1 - q}),$$

where m is the number of edges of the graph G.

In subsection 4.2.5 we will define a chain complex $\mathcal{D}(G)$ of doubly graded modules whose doubly graded Euler characteristic is equal to $D_G(t, q)$.

4.2.4 The categorification of the chromatic polynomial

Let n denote the number of vertices of the graph G. Let R be the ring of polynomials in n variables over $\mathbb{Q}$, i.e. $R = \mathbb{Q}[x_1, \ldots, x_n]$. We introduce a grading in R, by giving the degree 1 to every x_i. Order the set of vertices of G and to the i-th vertex assign the variable x_i. Finally, to every edge $e \in E(G)$, whose endpoints are the vertices i_e and j_e, assign the monomial $m_e = x_{i_e} - x_{j_e}$ (the ambiguity of the sign does not affect the later construction).

The cubic complex construction

Let $s \subset E(G)$ be a subset of the set of edges of G, and let $[G : s]$ be the corresponding state in the resolution of a graph G. Define the ideal I_s as the ideal generated by the monomials m_e, for all edges $e \in s$. Finally, to the state $[G : s]$ assign the module $R_s = R/I_s$.

Proposition 6 *The quantum graded dimension of R_s is equal to $(1-q)^{-k(s)}$, where $k(s)$ denotes the number of connected components of $[G : s]$.*

Proof:
Let i and j be two arbitrary vertices of G. They obviously belong to the same connected component of $[G : s]$ if and only if there exist a sequence of edges belonging to s which connects i and j, which obviously happens if and only if $x_i - x_j$ belongs to I_s. Hence, all the variables corresponding to the vertices from the same component, must be the same in R_s. In other words, R_s is isomorphic to the ring of polynomials (over $\mathbb{Q}$) in $k(s)$ variables, and hence:

$$q \dim R_s = \left(\sum_{i \geq 0} q^i \right)^{k(s)} = (1-q)^{-k(s)}.$$

■

Denote by m the number of edges of G, and fix an ordering on the set $E(G)$, denoted by $(e_1, \ldots, e_m)$. Now we will define the chain complex $\mathcal{C}$ in a standard way, by "summing over columns" of our cubic complex: for each i, with $0 \leq i \leq m$, we will define the i-th chain group, $\mathcal{C}^i(G)$ as the direct sum of R_s, over all $s \subset E(G)$, such that $|s| = i$.

Now, let us turn to the differential. We will define the map d^i from $\mathcal{C}^i(G)$ to $\mathcal{C}^{i+1}(G)$ as a sum of maps between the direct summands of the chain groups. The only nonzero maps are the maps from R_s to $R_{s\cup\{e\}}$, with

$e \notin s$ (which are exactly the ones that correspond to the edges of the cube), and we set them (up to a sign) to be the identity (i.e. the map that sends $f + I_s$ to $f + I_{s\cup\{e\}}$ for every $f \in R$).

We now introduce signs in a standard way in order to make the cube anticommutative, and hence to make the square of the differential equal to 0. Namely, we put minus signs exactly for those maps $R_s \to R_{s\cup\{e\}}$, with an odd number of edges in s which are ordered before e.

In this way we have obtained a chain complex, $\mathcal{C}(G)$, of graded R-modules with grading preserving differential. Its homology groups obviously don't depend of the ordering of the vertices, and also don't depend of the ordering of the edges of G (like in [15], section 2.2.3), and hence we obtain

Theorem 16 *The homology groups of the chain complex $\mathcal{C}(G)$ are invariants of the graph G, and the graded Euler characteristic of $\mathcal{C}(G)$ is equal to the chromatic polynomial $P_G(q)$.*

Alternative description

Now we will give an equivalent definition of the chain complex $\mathcal{C}(G)$ in terms of Koszul complexes.

To each edge $e \in E(G)$ we assign two complexes, $\mathcal{C}_{e-}$ and $\mathcal{C}_{e+}$ defined in the following way:

$$\mathcal{C}_{e-}: \quad 0 \longrightarrow R \xrightarrow{0} R \longrightarrow 0,$$

$$\mathcal{C}_{e+}: \quad 0 \longrightarrow R \xrightarrow{x_i - x_j} R \longrightarrow 0,$$

where i and j are the vertices of G which are the endpoints of the edge e. Now, to every subset $s \subset E(G)$ we assign a complex $\mathcal{C}_s$ which is the tensor product of $\mathcal{C}_{e\pm}$, where we take $+$ if $e \in s$ and $-$ if $e \notin s$. Finally, to the state $[G : s]$ we assign the cohomology of $\mathcal{C}_s$ at the rightmost position.

To build the differentials, we introduce the (grading preserving) maps d_e, as the maps induced on the cohomology by the following homomorphism from $\mathcal{C}_{e-}$ to $\mathcal{C}_{e+}$:

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "0"] \arrow[d, "0"'] & R \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & R \arrow[r, "x_i - x_j"] & R \arrow[r] & 0. \end{tikzcd} \tag{4.11}$$

Here we put the upper row in cohomological degree 0, and the lower one in cohomological degree 1.

Now, in order to define the differentials, just tensor all the chain complexes and maps between them from (4.11) over all edges e of $E(G)$. If we take the cohomology only at the rightmost position in each "horizontal" complex (the ones in the same cohomological degree with respect to the definition after (4.11)), and as the differentials are the induced maps between them, we obtain a complex $\mathcal{C}'(G)$ which is isomorphic to the complex $\mathcal{C}(G)$ from the previous subsection.

4.2.5 The categorification of the dichromatic polynomial

In order to categorify the dichromatic polynomial we will have to introduce a new grading direction, and we will use the whole Koszul complex (actually a slightly modified one) that we have used in the previous subsection.

We order the vertices of G, and to the i-th one ($1 \leq i \leq n = \sharp V(G)$), we assign the variable x_i. We define the bidegree of all x_i as (0,1). Define the bigraded ring R by $R = \mathbb{Q}[x_1, \ldots, x_n]$, where we put the field $\mathbb{Q}$ in bidegree (0,0).

To each edge e, such that its endpoints are the i-th and j-th vertex, we can associate two resolutions of the graph G: the first one with the edge e contracted (i.e. when we identify the vertices i and j), and the second one with the edge e deleted. To the first resolution we assign the following complex (denoted by $\mathcal{D}(e+)$):

$$\mathcal{D}(e+): \qquad 0 \longrightarrow R\{-1,1\} \xrightarrow{x_i - x_j} R \longrightarrow 0,$$

and to the second one we assign the complex $\mathcal{D}(e-)$ given by:

$$\mathcal{D}(e-): \qquad 0 \longrightarrow R\{-1,1\} \xrightarrow{0} R \longrightarrow 0.$$

Let s be an arbitrary subset of $E(G)$ and let $[G : s]$ be the corresponding state of G. Then, to that state we assign the (Koszul) complex $\mathcal{D}'(s)$ of bigraded R-modules obtained by tensoring the complexes $\mathcal{D}(e+)$, where e runs over all edges in s, and $\mathcal{D}(f-)$, where f runs over all edges in $E(G) \setminus s$. We denote its (bigraded) cohomology by $H'(s)$ (the direct sum of the cohomology groups of $\mathcal{D}'(s)$).

Proposition 7 *The quantum bigraded dimension of $H'(s)$ is equal to:*

$$(1 + t^{-1}q)^{m-n} \left(\frac{1 + t^{-1}q}{1 - q} \right)^{k(s)},$$

where $k(s)$ is the number of connected components of $[G : s]$, and n and m are the number of vertices and edges of G, respectively.

Proof:

Like in the proof of Proposition 6 we obtain that the cohomology at the rightmost position of $\mathcal{D}'(s)$ is isomorphic to the ring of polynomials in $k(s)$ variables. However, here we will also have the cohomology at the leftmost position in each of the $\mathcal{D}(e\pm)$, which is isomorphic to the same ring of polynomials in $k(s)$ variables, but shifted by the bidegree $\{-1,1\}$, for all $\mathcal{D}(e-)$ and for a certain number of the $D(e+)$. We will show by induction on $|s|$ that the total number of such e's, denoted by $c(s)$, is equal to $k(s)-n+m$.

If $|s|=0$ then we have the tensor product of m complexes with all the mappings equal to zero, and hence we have that the number of edges which contribute with nontrivial cohomology at the leftmost position is equal to $m=k(s)-n+m$ (note that in this case $k(s)=n$). Now suppose that the formula is true for some subset s and consider the state $[G:(s\cup e)]$ with $e\in E(G)\backslash s$. Denote the endpoints of e by i and j, and denote $s'=s\cup e$. This means that $\mathcal{D}'(s')$ is formed by the tensor product of the same complexes as $\mathcal{D}'(s)$ with $\mathcal{D}(e+)$ instead of $\mathcal{D}(e-)$. Now, $\mathcal{D}(e+)$ will have nontrivial cohomology at the leftmost position if and only if x_i-x_j belongs to the ideal generated by the monomials defined by the edges of s, i.e. if and only if the vertices i and j belong to the same connected component of $[G:s]$. In other words, we have $c(s')=c(s)$ if $k(s')=k(s)$ and $c(s')=c(s)-1$ if $k(s')=k(s)-1$. So we have $c(s)=k(s)-n+m$ as we wanted to prove.

Hence the total bigraded dimension of $H'(s)$ is equal to:

$$(1+t^{-1}q)^{k(s)-n+m}(1-q)^{-k(s)}.$$

■

Furthermore, to every vertex v of the graph G, we assign the same complex as $\mathcal{D}(e-)$. Now, if we tensor these complexes over all vertices of G and tensor the complex obtained with $\mathcal{D}'(s)$, we obtain the complex $\mathcal{D}(s)$. We denote the cohomology of $\mathcal{D}(s)$ by $H(s)$, and that is the space that we will assign to the state $[G:s]$. Obviously, we have:

$$q\dim H(s)=(1+t^{-1}q)^m\left(\frac{1+t^{-1}q}{1-q}\right)^{k(s)}.$$

In order to introduce the differentials between the cohomologies $H(s)$, we will define the (grading preserving) homomorphism $d(e)$ from $\mathcal{D}(e+)\{0,1\}$ to $\mathcal{D}(e-)$, and then for the differentials we take the induced mappings on cohomology. We define $d(e)$ by:

$$\begin{tikzcd}
0 \arrow[r] & R\{-1,2\} \arrow[r, "x_i - x_j"] \arrow[d, "x_i - x_j"'] & R\{0,1\} \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & R\{-1,1\} \arrow[r, "0"] & R \arrow[r] & 0.
\end{tikzcd}$$

We put the upper row in cohomological degree -1, and the lower row in cohomological degree 0. We will denote this complex of complexes by $\mathcal{D}_e$.

For the graph G define the complex of (Koszul) complexes by tensoring $\mathcal{D}_e$ over all edges e of G. By taking the cohomology $H^j(G)$, $-m \leq j \leq 0$, in each "horizontal" complex and defining the differentials between them to be the ones induced by the tensor product of $d(e)$'s, we obtain a triply graded complex $\mathcal{D}(G)$.

From the definition we have

Theorem 17 *The homotopy class of the complex $\mathcal{D}(G)$ is an invariant of the graph G whose bigraded Euler characteristic is equal to the dichromatic polynomial $D_G(t, q)$ of the graph G.*

Chapter 5

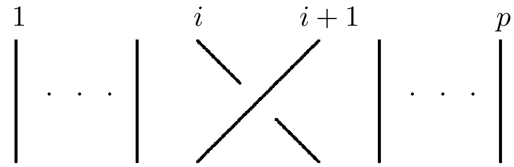
Properties of link homology for positive braid knots

5.1 Introduction

In this chapter we prove the conjecture from [28] that the first Khovanov homology group of the positive braid knot is trivial (see Theorem 18). In the proof we use the basic ingredients of the construction of the $sl(2)$ link homology (cubic complex, independence of the planar projection chosen), and so the major part can be directly applied to other link homology theories. Especially, in the case of $sl(n)$-link homology (see Section 2.6 of Chapter 2), whose particular definitions of chain groups and differentials make it practically incalculable, we modify slightly our proof to show that the first $sl(n)$ homology group of the positive braid knot is trivial for every positive integer n, as well.

5.2 Positive braid knots

The positive braid knots are the knots (or links) that are the closures of positive braids. Let K be an arbitrary positive braid knot and let D be its planar projection which is the closure of a positive braid. Denote the number of strands of that braid by p. We say that the crossing c of D is of the type σ_i, $i < p$, if it corresponds to the generator σ_i in the braid word of which D is the closure.



σ_i

Denote the number of crossings of the type σ_i by l_i, $i = 1, \ldots, p-1$ and order them from top to bottom. Then we can write each crossing c of D as a pair (i, α) (we will also write $(i\alpha)$ if there is no possibility of confusion), $i = 1, \ldots, p-1$ and $\alpha = 1, \ldots, l_i$, if c is of the type σ_i and it is ordered as α-th among the crossings of the type σ_i. Finally, we order the crossings of D by the following ordering: $c = (i\alpha) < d = (j\beta)$ if and only if $i < j$, or $i = j$ and $\alpha < \beta$.

Since a positive braid knot is a positive knot, we have that $\mathcal{H}^i(K) = 0$ for $i < 0$. Also if a positive braid knot K has a regular diagram D, which is the closure of the p-strand braid σ such that all letters σ_i, $i = 1, \ldots, p-1$, appear in the braid word of σ (i.e. D is not the disjoint union of two nonempty diagrams), we know that its zeroth homology group (see e.g. [28]) is two-dimensional (without torsion) and that the q-gradings of the two generators are $1 - p + n(D) \pm 1$, where $n(D)$ is the number of crossings of D.

5.3 The main result

In the following theorem we prove that the first homology group of a positive braid knot is trivial.

Theorem 18 *If K is a positive braid knot, then $\mathcal{H}^1(K)$ is trivial.*

Proof:

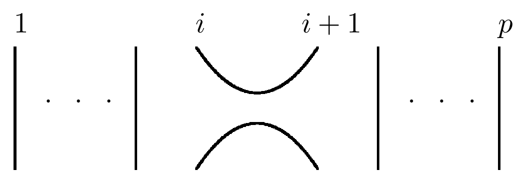
First of all, if D is a regular diagram of K, which is the closure of a positive braid, then we have that $\mathcal{H}^{i,j}(K) = H^{i,j-n(D)}(D)$, where $n(D)$ is the number of crossings of D, and so we have that $\mathcal{H}^1(K) = 0$ if and only if $H^1(D) = 0$. So we are going to show that the (unnormalized) first homology group of D is trivial.

In order to prove that $H^1(D) = 0$, we will use the definition of Khovanov homology (see Section 2.4 of Chapter 2), i.e. we will prove that for any element t' of the chain group $C^1(D)$ such that $d^1(t') = 0$, there exists an element $t \in C^0(D)$ such that $t' = d^0(t)$. For this we first need to understand the chain groups $C^0(D)$, $C^1(D)$ and $C^2(D)$ and the differentials d^0 and d^1.

$C^0(D)$ "comes" from all resolutions D_ϵ with $|\epsilon| = 0$. However, since D

is the closure of a positive braid, we have only one such resolution ϵ_0 (all crossings are resolved into 0-resolutions) and it is an unlink that consists of p unknots (i.e. the closure of the trivial braid with p strands). Hence, we have that $C^0(D) = V^{\otimes p}$ where we assigned the i-th copy of V (denoted by V^i) to the circle which is the closure of the i-th strand of D_{ϵ_0}.

Now, we pass to $C^1(D)$. It "comes" from all resolutions D_ϵ with $|\epsilon| = 1$, i.e. all resolutions where we resolve all except one crossing of D in a 0-resolution and the remaining one in a 1-resolution. In such a way, if the crossing c that is resolved into a 1-resolution is of the type σ_i (i.e. if $c = (i\alpha)$, for some $\alpha = 1, \ldots, l_i$), then the corresponding resolution, D_c, is the closure of the plat diagram E_i (see picture).



E_i

To that resolution we assign the vector space $V_c = V^{\otimes(p-1)}$, where we have assigned the first $i-1$ and the last $p-1-i$ copies of V to the circles that are the closures of the first $i-1$ and the last $p-i-1$ strands of the resolution D_c, respectively, and the i-th copy of V corresponds to the remaining circle (closure of E_i). So, we have that $C^1(D) = \bigoplus_{c \in c(D)} V_c\{1\}$. Further on, we denote the k-th copy of V in V_c by V_c^k.

The differential $d^0 : C^0(D) \to C^1(D)$ is given by the maps $f_c : V^{\otimes p} \to V_c$ where if c is of the type σ_i than f_c acts as the identity on the first $i-1$ and the last $p-i-1$ copies of V and as the multiplication m on the remaining two copies of V. In other words, f_c maps the copies V^j as the identity onto V_c^j, for $j < i$, maps the copies V^{j+1} as the identity onto V_c^j, for $i < j < p$, and on the remaining two factors f_c acts as the multiplication $m : V^i \otimes V^{i+1} \to V_c^i$.

Furthermore, $C^2(D)$ comes from the resolutions where exactly two of the crossings are resolved into 1-resolutions and the remaining ones are resolved into 0-resolutions. Denote the two crossings that are resolved into 1-resolutions by c and d (where c is ordered before d), and let c be of the type σ_i and d of the type σ_j. Then we have that $i \leq j$.

If $i = j$ then the corresponding resolution $D_{c,d}$ is the closure of the plat diagram E_i^2, has p circles and hence the corresponding summand $V_{c,d}$ of

$C^2(D)$ is isomorphic to $V^{\otimes p}$. Here we have assigned the first $i-1$ and the last $p-i-1$ copies of V to the circles that are the closures of the first $i-1$ and the last $p-i-1$ strands of the resolution $D_{c,d}$, respectively, while the i-th and $(i+1)$-th copy of V correspond to the remaining two circles that are formed from the i-th and $(i+1)$-th strand (the i-th copy of V corresponds to the outer, and the $(i+1)$-th copy of V to the inner circle).

If $i < j$ then the corresponding resolution is the closure of the plat diagram E_iE_j (or E_jE_i), has $p-2$ circles and hence the corresponding summand $V_{c,d}$ of $C^2(D)$ is isomorphic to $V^{\otimes(p-2)}$.

If $i+1 < j$, we assign the first $i-1$ copies of V to the closures of the first $i-1$ strands. The i-th copy of V is assigned to the circle that is obtained by joining the i-th and $(i+1)$-th strand (the 1-resolution of c). The following $j-i-2$ copies of V are assigned to the closures of the strands from the $(i+2)$-th to the $(j-1)$-th of $D_{c,d}$, respectively. The $(j-1)$-th copy of V is assigned to the circle that is obtained by joining the j-th and $(j+1)$-th strand (the 1-resolution of d). The remaining $p-j-1$ copies of V are assigned to the closures of the last $p-j-1$ strands of $D_{c,d}$.

If $i+1 = j$, then we assign the first $i-1$ copies of V to the closures of the first $i-1$ strands. The i-th copy of V is assigned to the circle that is obtained by joining the i-th, $(i+1)$-th and $(i+2)$-th strand (the 1-resolutions of c and d). The remaining $p-i-2$ copies of V are assigned to the closures of the strands from the $(i+3)$-th to the p-th of $D_{c,d}$, respectively.

In all previous cases, we denote the k-th copy of V in $V_{c,d}$ by $V_{c,d}^k$.

Finally, the second chain group is

$$C^2(D) = \bigoplus_{c,d\in c(D),\, c<d} V_{c,d}\{2\}.$$

Now, we can describe the differential $d^1 : C^1(D) \to C^2(D)$. It is given as the sum of the maps of the form $f_{ecd} : V_e \to V_{c,d}$ (with $c, d, e \in c(D)$, $c < d$), where f_{ecd} is zero unless $e = c$ or $e = d$. Let $c = (i\alpha)$ and $d = (j\beta)$, for some $1 \le i \le j \le p-1$, $\alpha = 1, \dots, l_i$ and $\beta = 1, \dots, l_j$. The maps $f_{ccd} : V_c \to V_{c,d}$ are as follows:

If $i = j$ then f_{ccd} is given by the identity maps $id : V_c^l \to V_{c,d}^l$, for $l < i$, and $id : V_c^l \to V_{c,d}^{l+1}$, for $i < l < p$, and by the comultiplication map $\Delta : V_c^i \to V_{c,d}^i \otimes V_{c,d}^{i+1}$.

If $i < j$, then f_{ccd} is given by the identity maps $id : V_c^l \to V_{c,d}^l$, for $l < j-1$, and $id : V_c^l \to V_{c,d}^{l-1}$, for $j < l \le p-1$, and by the multiplication map $m : V_c^{j-1} \otimes V_c^j \to V_{c,d}^{j-1}$.

The other class of nonzero maps $f_{dcd} : V_d \to V_{c,d}$ is in the case $i = j$ given by $-f_{ccd}$ (because of the signs $(-1)^{f_\nu}$ sprinkled around the cubic complex and the fact that in this case $c < d$ and $V_c = V_d$). In the case $i < j$, f_{dcd} is given as $-g_{cd}$ (the minus sign is because of the same reasons as above), where g_{cd} is given by the identity maps: $id : V_d^l \to V_{c,d}^l$, for $l < i$, and $id : V_d^{l+1} \to V_{c,d}^l$, for $i < l < p-1$, and by the multiplication map $m : V_d^i \otimes V_d^{i+1} \to V_{c,d}^i$.

Now, we can go back to the proof. Let

$$t' = (t_{1,1}, \dots, t_{1,l_1}, t_{2,1}, \dots, t_{2,l_2}, \dots, t_{p-1,1}, \dots, t_{p-1,l_{p-1}}) \in C^1(D)$$

be such that $d^1(t') = 0$. Here we have that $t_{i,\alpha} \in V_{(i\alpha)}$ for $i = 1, \dots, p-1$, $\alpha = 1, \dots, l_i$. We shall also write $t_{i\alpha}$ for $t_{i,\alpha}$ if there is no possibility of confusion. Our aim is to find an element $t \in C^0 = V^{\otimes p}$ such that $d^0(t) = t'$.

Since $d^1(t') = 0$ we have that its projection, denoted by $d^1_{i\alpha\beta}(t')$, to the space $V_{(i\alpha),(i\beta)}$ is equal to zero, for every $i = 1, \dots, p-1$, and $\alpha, \beta = 1, \dots, l_i$. However, this implies that $t_{i\alpha} = t_{i\beta}$ for every $i = 1, \dots, p-1$, $\alpha, \beta = 1, \dots, l_i$, since only the maps from $V_{(i\alpha)}$ and $V_{(i\beta)}$ to $V_{(i\alpha),(i\beta)}$ are nonzero, and the only nonidentity part of the mappings is the comultiplication Δ on the same (i-th) copy of V in both $V_{(i\alpha)}$ and $V_{(i\beta)}$.

Hence, we have obtained that $t' \in \ker d^1$ if and only if $t_{i\alpha} = t_{i\beta}$ for every $i = 1, \dots, p-1$, $\alpha, \beta = 1, \dots, l_i$ and $\bar{t} = (t_{1,1}, t_{2,1}, \dots, t_{p-1,1}) \in \ker \bar{d}^1$, where $\bar{d}^1$ is the restriction of d^1 to $W_1 = V_{(1,1)} \oplus V_{(2,1)} \oplus \cdots V_{(p-1,1)}$ (we omit $V_{(i,1)}$ if there are no crossings of the type σ_i in D), composed with the projection onto $W_2 = \bigoplus_{1 \le i < j \le p-1} V_{(i1),(j1)}$ (omitting the indices i, j of crossings which do not appear in D).

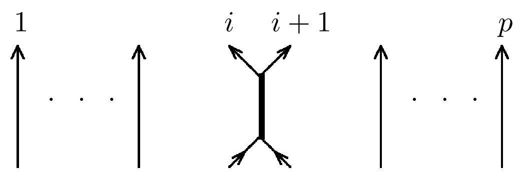
Furthermore, note that for every $i = 1, \dots, p-1$, and $\alpha, \beta = 1, \dots, l_i$, the projection of $d^0(t)$, for any $t \in C^0(D)$ to $V_{(i\alpha)}$ and $V_{(i\beta)}$ is equal, i.e. the differential d^0 is completely determined by the map $\bar{d}^0$ which is the projection of d^0 on W_1. Finally, we have that if there exists $t \in C^0$ such that $\bar{d}^0(t) = \bar{t}$, then $d^0(t) = t'$. Hence, to finish the proof, we are left with proving that for every $y \in \ker \bar{d}^1 \subset W_1$, there exists $x \in C^0(D)$, such that $\bar{d}^0(x) = y$.

Now, observe the positive braid knot K' which has the regular diagram D' which is the closure of the braid $\sigma_1 \sigma_2 \cdots \sigma_{p-1}$ (where we omit the symbols σ_i which are not contained in the braid word of which D is the closure). Its zeroth chain group $C^0(D')$ is obviously equal to $C^0(D) = V^{\otimes p}$, its first chain group $C^1(D')$ is equal to W_1 and its second chain group $C^2(D')$ is equal to W_2. Its zeroth differential is equal to the previously defined $\bar{d}^0$,

while its first differential is equal to $\bar{d}^1$. Since K' is isotopic to the unknot (or to an unlink consisting of unknots), its first homology group is trivial and hence for every $y \in \ker \bar{d}^1$ there exists $x \in C^0(D') = C^0(D)$ such that $\bar{d}^0(x) = y$. This concludes the proof. ■

5.4 The $sl(n)$ case

In this section we will adapt our proof from the previous section to show that the first $sl(n)$ homology group of a positive braid knot is trivial. As we saw in Section 2.6, the basic principle of the construction, namely the cubic complex, 0- and 1-resolutions of the crossings, per-edge maps and summing of the columns are the same as in the $sl(2)$ case (Section 2.4 and the previous section). The slight differences in the $sl(n)$ case for positive braid knots are: first of all, the 1-resolution of the crossing $c = (i\alpha)$ is given by the thick edge resolution $\bar{E}_i$:



$\bar{E}_i$

instead of the plat E_i. Hence the total resolutions, D_ϵ, are in this case trivalent graphs with thick edges. We denote the total resolution when all crossings of D are resolved into 0-resolution by D_0, the specific total resolution with the crossing c resolved into a 1-resolution and all other crossings into 0-resolutions by D_c, and the specific total resolution with the crossings c and d resolved into 1-resolutions and all other crossings into 0-resolutions by $D_{c,d}$ (like in the $sl(2)$ case). Furthermore, we assign to the diagram D_ϵ the $\mathbb{Q}$-vector space $\bar{V}_\epsilon$ (we will also write $\bar{V}(D_\epsilon)$), and we define the i-th chain group by

$$C_n^i(D) = \oplus_{|\epsilon|=i} \bar{V}_\epsilon\{-i\}.$$

Also, we define the differentials as the sum (we have already included the same signs as in the $sl(2)$ case) of the grading preserving per-edge maps $\bar{d}_\nu : \bar{V}_\epsilon \to \bar{V}_{\epsilon'}\{-1\}$. Again, we denote $\bar{d}_\nu : \bar{V}_0 \to \bar{V}_c\{-1\}$ by f_c, and $\bar{d}_\nu : \bar{V}_e \to \bar{V}_{c,d}\{-1\}$ by f_{ecd}. Since we sum only per-edge maps, f_{ecd} is nonzero only if $e = c$ or $e = d$.

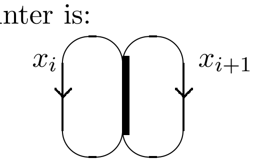
The main difference in the definition of $sl(n)$ homology compared to the $sl(2)$ case, is the very complicated assignment of the vector spaces $\bar{V}_\epsilon$ as well

as the maps $\bar{d}_\nu$, which make the theory practically incalculable, and very difficult to deal with. However, in the case that we are interested in, we only need the chain groups $C_n^0(D)$, $C_n^1(D)$ and $C_n^2(D)$, and the differentials d_n^0 and d_n^1, and their definitions can be extracted explicitly from the definition (see Section 2.6.2 and also [30]).

First of all, if we have two disjoint trivalent graphs with wide edges, Γ_1 and Γ_2, then the vector space assigned to their disjoint union, $\bar{V}(\Gamma_1 \coprod \Gamma_2)$, is isomorphic to the tensor product of the spaces $\bar{V}(\Gamma_1)$ and $\bar{V}(\Gamma_2)$. Next, if we have a braid diagram with p strands, we assign the variable x_i to the i-th strand. We introduce a grading on the ring of polynomials $R = \mathbb{Q}[x_1, \ldots, x_p]$, by defining $\deg x_i = 2$. To the circle (unknot) labeled by the variable x_i, we assign the vector space A_i given by:

$$A_i = \mathbb{Q}[x_i]/(x_i^n)\{1-n\}.$$

Note that the quantum dimension of A_i is equal to $[n]$. Another type of diagrams that we will encounter is:



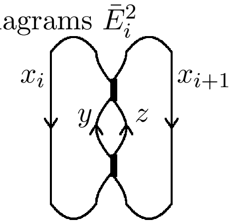
i.e. the closure of $\bar{E}_i$ (below, by $\bar{E}_i$ we mean only the part of the whole braid formed by the strands that are connected to a wide edge). To this graph we assign the vector space B_i given by:

$$B_i = \mathbb{Q}[x_i, x_{i+1}]/(\pi_{i,i+1}^n, \pi_{i,i+1}^{n-1})\{3-2n\}.$$

Here, by $\pi_{i,j}^k$ we denote the following polynomial:

$$\pi_{i,j}^k = \frac{x_i^{k+1} - x_j^{k+1}}{x_i - x_j} = \sum_{l=0}^{k} x_i^l x_j^{k-l}.$$

Next, to the closure of the diagrams $\bar{E}_i^2$



we assign the graded vector space C_i, defined by:

$$C_i = \mathbb{Q}[x_i, x_{i+1}, y]/(\pi_{i,i+1}^n, \pi_{i,i+1}^{n-1}, (y - x_i)(y - x_{i+1}))\{2-2n\},$$

where y is a new variable with $\deg y = 2$.

To the graph which is the closure of the diagram $\bar{E}_i\bar{E}_j$, for $|i-j|>1$ we assign the vector space $F_{i,j}$ defined by

$$F_{i,j} = \mathbb{Q}[x_i, x_{i+1}, x_j, x_{j+1}]/(\pi^n_{i,i+1}, \pi^{n-1}_{i,i+1}, \pi^n_{j,j+1}, \pi^{n-1}_{j,j+1})\{6-4n\}.$$

Finally, to the closure of $\bar{E}_i\bar{E}_{i+1}$ we assign the vector space $F_{i,i+1}$ defined by

$$F_{i,i+1} = \mathbb{Q}[x_i, x_{i+1}, x_{i+2}]/(\pi^n_{i,i+1}, \pi^{n-1}_{i,i+1}, \pi^n_{i+1,i+2}, \pi^{n-1}_{i+1,i+2})\{5-3n\}.$$

Now we can give the main theorem:

Theorem 19 *For every positive braid knot K, we have that the homology group $\mathcal{H}^1_n(K)$ is trivial.*

Proof:

Mainly we will adapt our proof from the previous Section for this case. First of all, we again take a diagram D of K, which is the closure of the positive braid σ. Denote the number of strands of σ by p, and assign the variable x_i, with $\deg x_i = 2$, to the i-th strand. Since D has only positive crossings we have that $\mathcal{H}^1_n(K)$ is trivial if and only if $H^1_n(D)$ is trivial. So, we have to prove that the latter group is trivial.

Again, we use the direct sum definition of the chain groups $C^0_n(D)$, $C^1_n(D)$, $C^2_n(D)$, and of the differentials d^0_n and d^1_n. We will prove that for each $t' \in C^1_n(D) = \bigoplus_c \bar{V}_c\{-1\}$ such that $d^1_n(t') = 0$, there exists $t \in C^0_n(D) = \bar{V}_0 = \bigotimes_{i=1}^p A_i$ such that $d^0_n(t) = t'$.

Let $i \in \{1, \ldots, p-1\}$ and $1 \le \alpha < \beta \le l_i$ be arbitrary. Denote $c = (i\alpha)$ and $d = (i\beta)$. We denote the restriction of t' to the space $\bar{V}_c = \bar{V}_{(i\alpha)}$ by $t_{i\alpha}$. Again the differential d^1_n maps only the spaces $\bar{V}_{(i\alpha)}$ and $\bar{V}_{(i\beta)}$ into $\bar{V}_{(i\alpha),(i\beta)}\{-1\}$. Let us see what these spaces and maps are.

The space $\bar{V}_c = \bar{V}_{(i\alpha)}$ is assigned to the diagram D_c which is the disjoint union of $p-2$ circles and the closure of the diagram $\bar{E}_i$. Hence, we have:

$$\bar{V}_c = A_1 \otimes \cdots \otimes A_{i-1} \otimes B_i \otimes A_{i+2} \otimes \cdots \otimes A_p,$$

and we have completely the same expression for $\bar{V}_d = \bar{V}_{(i\beta)}$. The space $\bar{V}_{c,d} = \bar{V}_{(i\alpha),(i\beta)}$ is assigned to the diagram $D_{c,d}$ which is the disjoint union of $p-2$ circles and the closure of the diagram $\bar{E}_i^2$, and so:

$$\bar{V}_{(i\alpha),(i\beta)} = A_1 \otimes \cdots \otimes A_{i-1} \otimes C_i \otimes A_{i+2} \otimes \cdots \otimes A_p.$$

Finally, the map $f_{ccd} : \bar{V}_c \to \bar{V}_{c,d}$ is the identity on the factors A_j, for $1 \le j < i$ and $i+1 < j \le p$, and on the remaining part is given as the multiplication by $y - x_{i+1}$. The map $f_{dcd} : \bar{V}_d \to \bar{V}_{c,d}$ is just the negative of f_{ccd} (i.e. $f_{ccd} = -f_{dcd}$), because of the signs sprinkled around the cube of resolutions that make it anticommutative.

Since $d_n^1(t') = 0$ we have that the projection of $d_n^1(t')$ to $\bar{V}_{c,d}$ is zero. However, the only nonzero maps that have $\bar{V}_{c,d}$ as the codomain are f_{ccd} and f_{dcd}, and so we have that $t_{i\alpha} = t_{i\beta}$, like in the $sl(2)$ case.

Furthermore, if $c = (i\alpha)$ and $d = (j\beta)$, for $i < j$, then the vector spaces $\bar{V}_{c,d}$ do not depend on α and β (they are the tensor product of $F_{i,j}$ and A_k's), and the maps f_{ccd} and f_{dcd} are, up to a sign, identities (multiplications by 1). Also, for all crossings $c = (i\alpha)$, the spaces $\bar{V}_c$ do not depend on α (they are the tensor product of B_i and A_k's), and the maps f_c are identities.

Hence, like in the $sl(2)$ case, we have that $d_n^1(t') = 0$ if and only if $t_{i\alpha} = t_{i\beta}$ for every $i = 1, \ldots, p-1$, $\alpha, \beta = 1, \ldots, l_i$ and $\bar{t} = (t_{1,1}, t_{2,1}, \ldots, t_{p-1,1}) \in \ker \bar{d}^1$, where $\bar{d}^1$ is the restriction of d^1 to $\bar{W}_1 = \bar{V}_{(1,1)} \oplus \bar{V}_{(2,1)} \oplus \cdots \bar{V}_{(p-1,1)}$ (here we have omitted $\bar{V}_{(i,1)}$ if there are no crossings of type σ_i in D) composed with the projection onto $\bar{W}_2 = \bigoplus_{1 \le i < j \le p-1} \bar{V}_{(i1),(j1)}$ (omitting the indices i, j of crossings which do not appear in D). Also, for every $i = 1, \ldots, p-1$, $\alpha, \beta = 1, \ldots, l_i$ and $t \in C_n^0(D)$, the projection of $d_n^0(t)$ to $\bar{V}_{(i\alpha)}$ and $\bar{V}_{(i\beta)}$ is equal, i.e. the differential d_n^0 is completely determined by the map $\bar{d}_n^0$ which is the projection of d_n^0 onto $\bar{W}_1$. So, we have that if there exists $t \in C_n^0(D)$ such that $\bar{d}_n^0(t) = \bar{t}$, then $d_n^0(t) = t'$. Hence, to finish the proof, we are left with proving that for every $y \in \ker \bar{d}_n^1 \subset \bar{W}_1$, there exists $x \in C_n^0(D)$, such that $\bar{d}_n^0(x) = y$.

However, the last statement again follows from observing the positive braid knot K' which has the regular diagram D' which is the closure of the braid $\sigma_1\sigma_2 \cdots \sigma_{p-1}$ (where we omit the symbols σ_i which are not contained in the braid word of which D is the closure). Its zeroth chain group $C_n^0(D')$ is obviously equal to $C_n^0(D)$, its first chain group $C_n^1(D')$ is equal to $\bar{W}_1$ and its second chain group $C^2(D')$ is equal to $\bar{W}_2$. Its zeroth differential is equal to the previously defined $\bar{d}_n^0$, while its first differential is equal to $\bar{d}_n^1$. Since K' is isotopic to the unknot (or to an unlink consisting of unknots), its first homology group is trivial (from Theorem 3) and hence for every $y \in \ker \bar{d}_n^1$ there exists $x \in C_n^0(D') = C_n^0(D)$ such that $\bar{d}_n^0(x) = y$. This concludes the proof. ■

Chapter 6

Thickness and stability of link homology for torus knots

6.1 Introduction

In this chapter we first show that the torus knots $T_{p,q}$ for $3 \leq p \leq q$ (non-alternating torus knots) are homologically thick, i.e. that their Khovanov homology occupies at least three diagonals. Furthermore, in the course of the proof we obtain even stronger results that relate the homology of the torus knots $T_{p,q}$ and $T_{p,q+1}$. Namely, we prove that, up to a certain homological degree, their (unnormalized) homologies coincide.

As the first application of this result we calculate the homology of torus knots for low homological degrees. We also obtain the proof of the existence of stable Khovanov homology of torus knots, conjectured in [11].

Furthermore, we conjecture that the homological width of the torus knot $T_{p,q}$ is at least p, and we reduce this problem to determining the nontriviality of certain homological groups.

Since in the proofs we mainly use the long exact sequence of Khovanov homology (2.5) – we do not rely heavily on the explicit definition of $sl(2)$ homology – we also obtain most of the analogous results for the stability of $sl(n)$ homology of torus knots.

6.2 Torus knots

A knot or a link is a torus knot if it is isotopic to a knot or a link that can be drawn without any points of intersection on the trivial torus. Every torus link is, up to a mirror image, determined by two nonnegative integers p and

q, i.e. it is isotopic to a unique torus knot $T_{p,q}$ which has the diagram $D_{p,q}$ - the closure of the braid $(\sigma_1\sigma_2\ldots\sigma_{p-1})^q$ - as a planar projection. In other words, $D_{p,q}$ is the closure of the p-strand braid with q full twists. Since $T_{p,q}$ is isotopic to $T_{q,p}$ we can assume that $p \le q$.

The torus knots are obviously positive braid knots (since the diagram $D_{p,q}$ is the closure of a positive braid). Hence, their homology satisfies the properties from the previous chapter. Also, we will use the same notation and conventions from Section 5.2.

6.3 Thickness of torus knots

If $p = 1$ then the torus knot $T_{p,q}$ is trivial and for $p = 2$ the torus knot $T_{2,q}$ is alternating, hence its homology occupies exactly two diagonals. However, if $p \ge 3$, the torus knot $T_{p,q}$ is non-alternating and we will prove that its homology occupies at least three diagonals. Namely, we prove the following theorem:

Theorem 20 *Let $K = T_{p,q}$, $3 \le p \le q$ be a torus knot. Then*

$$\operatorname{rank} \mathcal{H}^{4,(p-1)(q-1)+5}(K) > 0.$$

From this theorem we obtain

Corollary 21 *Every torus knot $T_{p,q}$, $p, q \ge 3$ is H-thick, i.e. its Khovanov homology occupies at least three diagonals.*

Proof (of Corollary 21):

Since $T_{p,q}$ is a positive knot, its zeroth homology group is two dimensional and the q-gradings (and consequently the δ-gradings) of its two generators are $(p-1)(q-1)-1$ and $(p-1)(q-1)+1$, respectively (see e.g. Section 5.2). However, from Theorem 20 we have that there exists a generator with t-grading equal to 4 and q-grading equal to $(p-1)(q-1)+5$, and so its δ-grading is equal to $(p-1)(q-1)+5-2\cdot 4 = (p-1)(q-1)-3$. Thus, we have obtained three generators of the homology of the torus knot $T_{p,q}$ which have three different values of the δ-grading and hence its Khovanov homology occupies at least three diagonals. ■

Now we give a proof of Theorem 20.

Proof:

First of all, since $K = T_{p,q}$ is a positive braid knot whose regular diagram $D_{p,q}$ is the closure of the braid $(\sigma_1\sigma_2\ldots\sigma_{p-1})^q$ with $(p-1)q$ crossings, we have that $\mathcal{H}^{4,(p-1)(q-1)+5}(K) = H^{4,6-p}(D_{p,q})$. So, we will "concentrate" on calculating the latter homology group, i.e. showing that its rank is nonzero. In order to do this we will use the long exact sequence (2.5) and we will relate the unnormalized fourth homology groups of the standard regular diagrams of the torus knots $D_{p,q}$ and $D_{p,q-1}$ for $p < q$.

Let $3 \leq p < q$. Let c_{p-1} be the crossing $(p-1,1)$ of the diagram $D_{p,q}$. Now denote by $E^1_{p,q}$ and $D^1_{p,q}$ the 1- and 0-resolutions, respectively, of the diagram $D_{p,q}$ at the crossing c_{p-1}. Then from (2.5) we obtain

$$\cdots \to H^{3,j-1}(E^1_{p,q}) \to H^{4,j}(D_{p,q}) \to H^{4,j}(D^1_{p,q}) \to H^{4,j-1}(E^1_{p,q}) \to H^{5,j}(D_{p,q}) \to \cdots$$

Now, we can continue the process, and resolve the crossing $c_{p-2} = (p-2,1)$ of $D^1_{p,q}$ in two possible ways. Denote the diagram obtained by the 1-resolution by $E^2_{p,q}$, and the diagram obtained by the 0-resolution by $D^2_{p,q}$. Then from the long exact sequence (2.5) we have:

$$\cdots \to H^{3,j-1}(E^2_{p,q}) \to H^{4,j}(D^1_{p,q}) \to H^{4,j}(D^2_{p,q}) \to H^{4,j-1}(E^2_{p,q}) \to H^{5,j}(D^1_{p,q}) \to \cdots$$

After repeating this process $p-1$ times (resolving the crossing $c_{p-k} = (p-k,1)$, $k = 1,\ldots,p-1$, of $D^{k-1}_{p,q}$, obtaining the 1-resolution $E^k_{p,q}$ and 0-resolution $D^k_{p,q}$ and applying the same long exact sequence in homology), we obtain that for every $i = 1,\ldots,p-1$:

$$\cdots \to H^{3,j-1}(E^i_{p,q}) \to H^{4,j}(D^{i-1}_{p,q}) \to H^{4,j}(D^i_{p,q}) \to H^{4,j-1}(E^i_{p,q}) \to H^{5,j}(D^{i-1}_{p,q}) \to \cdots \tag{6.1}$$

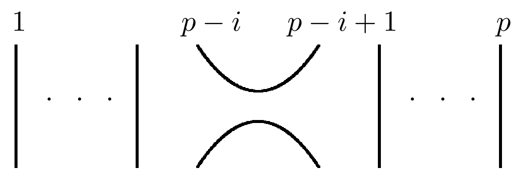
Here $D^0_{p,q}$ denotes $D_{p,q}$, and we obviously have that $D^{p-1}_{p,q} = D_{p,q-1}$.

Our goal is to show $H^3(E^i_{p,q})$ and $H^4(E^i_{p,q})$ are trivial for every $0 < i < p$. This is done in the following lemma.

Lemma 1 *For every three positive integers p, q and i, such that $3 \leq p < q$ and $i < p$, the knot with the diagram $E^i_{p,q}$ is positive, and the diagram $E^i_{p,q}$ has at least $p+q-3$ negative crossings.*

Proof:

Since for every $0 < i < p$, $E^i_{p,q}$ is obtained by the 1-resolution of the crossing $c_{p-i} = (p-i,1)$ of the (positive braid knot) diagram $D^{i-1}_{p,q}$, it is the closure of the plat braid diagram with only one plat E_{p-i}:



E_{p-i}

Now, note that two lower strands of E_{p-i} are always "neighbour" strands, i.e. they form a ribbon, through the diagram, until they reach the upper part of E_{p-i}. So, we can"slide" the lower part of the plat E_{p-i} through the diagram (by using the second Reidemeister move – R2 – and also the first Reidemeister move – R1 – where the two strands intersect each other) until it reaches the left or the right hand side of the upper part of E_{p-i}. If it first reaches the right hand side, it automatically (or after a R1 move) becomes the closure of a positive braid diagram. If it first reaches the left hand side then after a R1 move and a "slide" (sequence of R2 moves) we obviously obtain a positive braid diagram.

Concerning the number of negative crossings of $E^i_{p,q}$, note that by performing the first sequence of R2 moves (sliding the lower part of E_{p-i} through the diagram, from the top to the bottom) in each move we have "canceled" one positive and one negative crossing. Furthermore, since $p < q$, the two strands of the lower part of the plat E_{p-i} will make a full twist at least once, and so they will have two crossings with each other, which are both obviously negative crossings. So, we have that on each of the last (lower) $q-1$ blocks $(\sigma_1 \ldots \sigma_{p-1})$ of $E^i_{p,q}$ we have at least one negative crossing. Furthermore, on the part where both lower strands of the plat E_{p-i} make full twists, we have applied $p-2$ R2 moves, and hence we have in addition, at least, $p-2$ negative crossings.

Altogether, this gives at least $q-1+p-2 = p+q-3$ negative crossings of $E^i_{p,q}$, as required. ■

Now, we can go back to the proof. From the previous Lemma, we conclude that for every $0 < k < p$, $\mathcal{H}^{i,j}(E^k_{p,q})$ is trivial for $i < 0$, and that

$$H^{i,j}(E^k_{p,q}) \text{ is trivial for } i < p+q-3, \tag{6.2}$$

Hence, if $p+q-3 > 4$, from (6.1) and (6.2), we obtain that

$$H^{4,j}(D^{i-1}_{p,q}) = H^{4,j}(D^i_{p,q}), \quad i = 1, \ldots, p-1,$$

and thus we have that

$$H^{4,j}(D_{p,q}) = H^{4,j}(D_{p,q-1}), \quad \text{for } p+q > 7,\ p < q. \tag{6.3}$$

So, we can decrease the number of full twists, q, without changing the fourth homology group.

If $p = q = 3$, then e.g. by using programs for computing Khovanov homology ([7], [52]), we obtain that $\operatorname{rank} H^{4,3}(D_{3,3}) = \operatorname{rank} \mathcal{H}^{4,9}(T_{3,3}) = 1$, as wanted.

If $3 = p < q$, then from (6.3), we have that $H^{4,3}(D_{3,q}) = H^{4,3}(D_{3,4})$. However, by using programs for computing Khovanov homology, we obtain that the rank of the latter group (which is equal to $\mathcal{H}^{4,11}(T_{3,4})$) is equal to 1.

Remark 22 *In Bar-Natan's tables of knots in [5], the link $T_{3,3}$ is denoted by 6^3_3, and the torus knot $T_{3,4}$ is isotopic to the knot 8_{19}. For the general notation of knots and links see [47] and [7].*

Now, let us move to the general case $4 \le p \le q$. Then if $p < q$ we can apply (6.3) and obtain $H^{4,6-p}(D_{p,q}) = H^{4,6-p}(D_{p,p})$. Thus, we are left with proving that the latter homology group is of nonzero rank.

Now, apply the set of long exact sequences (6.1) for the case $p = q$. In this case, like in Lemma 1, we obtain that $E^k_{p,p}$ is the diagram of a positive knot, for every $k = 1, \ldots, p-1$. Furthermore, every diagram $E^k_{p,p}$ has exactly $2p-3$ negative crossings, and so we have:

$$H^i(E^k_{p,p}) \text{ is trivial for every } i < 2p-3. \tag{6.4}$$

Since $p > 3$, we have that $H^3(E^k_{p,p})$ and $H^4(E^k_{p,p})$ are trivial for all k, and so we have that

$$H^4(D_{p,p}) = H^4(D_{p,p-1}). \tag{6.5}$$

On the other hand, we have that

$$H^{4,6-p}(D_{p,p-1}) = \mathcal{H}^{4,p^2-3p+7}(T_{p,p-1}) = \mathcal{H}^{4,p^2-3p+7}(T_{p-1,p}) = H^{4,7-p}(D_{p-1,p}).$$

If $p > 4$, then we have from (6.3) that $H^{4,7-p}(D_{p-1,p}) = H^{4,7-p}(D_{p-1,p-1})$. By repeating this process, we can decrease the number of strands p, and obtain that:

$$H^{4,6-p}(D_{p,p}) = H^{4,2}(D_{4,4}).$$

Finally, from (6.5) we have

$$H^{4,2}(D_{4,4}) = H^{4,2}(D_{4,3}) = \mathcal{H}^{4,11}(T_{3,4}),$$

and the last homology group, as we saw previously, is of rank 1. This concludes our proof. ∎

6.4 Stability of Khovanov homology for torus knots

In the course of proving Theorem 20, apart from showing that the torus knots are H-thick, we have obtained some other properties of the homology of the torus knots. Namely, we proved that we can reduce the number of full twists, q, of the standard diagram $D_{p,q}$ of the torus knot $T_{p,q}$ without changing the first $p+q-3$ homology groups. In other words, we have obtained the existence of stable homology of torus knots (see Section 6.6 and [11]).

In the following theorem we summarize the stability properties obtained in the previous section.

Theorem 23 *Let p, q and i be integers such that $2 \le p < q$ and $i < p+q-3$. Then for every $j \in \mathbb{Z}$*

$$H^{i,j}(D_{p,q}) = H^{i,j}(D_{p,q-1}). \tag{6.6}$$

Furthermore, for every $2 \le p < q$ and $i < 2p-1$ and $j \in \mathbb{Z}$ we have

$$H^{i,j}(D_{p,p+1}) = H^{i,j}(D_{p,p+2}) = \cdots = H^{i,j}(D_{p,q}). \tag{6.7}$$

Also, for every $p \ge 2$, $i < 2p-3$ and $j \in \mathbb{Z}$, we have

$$H^{i,j}(D_{p,p}) = H^{i,j+1}(D_{p-1,p}). \tag{6.8}$$

Proof:

The equations (6.6) and (6.8) we have already obtained in the course of proving Theorem 20 (long exact sequences and formulas (6.2) and (6.4)). Formula (6.7) obviously follows from (6.6) since $i < 2p-1 = p+(p+2)-3$. ∎

The torus knots $T_{2,q}$ are alternating and their homology is well-known (see e.g. [24]). However, as the first corollary of the previous theorem we obtain the homology groups of $T_{p,q}$, for $3 \le p \le q$, with low homological degree.

Theorem 24 *Let $3 \le p \le q$ with p and q not both equal to 3. Then we have*

$$\begin{aligned}
\mathcal{H}^{0,(p-1)(q-1)\pm 1}(T_{p,q}) &= \mathbb{Z}\\
\mathcal{H}^{2,(p-1)(q-1)+3}(T_{p,q}) &= \mathbb{Z}\\
\mathcal{H}^{3,(p-1)(q-1)+7}(T_{p,q}) &= \mathbb{Z}\\
\mathcal{H}^{3,(p-1)(q-1)+5}(T_{p,q}) &= \mathbb{Z}_2\\
\mathcal{H}^{4,(p-1)(q-1)+6\pm 1}(T_{p,q}) &= \mathbb{Z}.
\end{aligned}$$

All other $\mathcal{H}^{i,j}(T_{p,q})$ for $i = 0, \ldots, 4$, are trivial.

Proof:

Suppose that $p = 3$. Then by applying (6.7), we obtain that $H^{i,j}(D_{3,q}) = H^{i,j}(D_{3,4})$ for $i = 0, \dots, 4$. If $p > 3$, then by applying (6.6) repeatedly, we obtain $H^{i,j}(D_{p,q}) = H^{i,j}(D_{p,p})$. Furthermore, by applying (6.8) (and then (6.6)) repeatedly we obtain $H^{i,j}(D_{p,p}) = H^{i,j+p-3}(D_{3,4})$ for $i = 0, 1, 2, 3, 4$. Finally, the homology of the last torus knot is well-known, see e.g. [53]-knot 8_{19}, and thus we obtain the required result. ∎

6.5 Further thickness results

Even though we have shown that the torus knots $T_{p,q}$, $p \geq 3$ are H-thick, from the existing experimental results one can see that the homology of torus $T_{p,q}$ knots occupies at least p diagonals (i.e. that its homological width is at least p). In fact, one can see that in all examples we have that $H^{2p-2,p}(D_{p,q})$ is of nonzero rank.

Proposition 8 *If* $\operatorname{rank} H^{2p-2,p}(D_{p,q}) > 0$ *then the homological width of the torus knot* $T_{p,q}$ *is at least* p*.*

Proof:

As we know, see e.g. the proof of Corollary 21, there exists a generator of the homology group $\mathcal{H}^{0,(p-1)(q-1)+1}$ and its δ-grading is equal to $(p-1)(q-1)+1$. Since we assumed that

$$\operatorname{rank} H^{2p-2,p}(D_{p,q}) = \operatorname{rank} \mathcal{H}^{2p-2,p+(p-1)q}(T_{p,q}) > 0$$

we have that there exists a generator of this homology group whose δ-grading is equal to $p + (p-1)q - 2(2p-2) = (p-1)(q-1) + 3 - 2p$. So, we have two generators whose δ-gradings differ by $2p-2$, and hence they lie on two different diagonals between which there are $p-2$ diagonals. Hence the homological width of the torus knot $T_{p,q}$ is at least p. ∎

Thus we are left with proving that $\operatorname{rank} H^{2p-2,p}(D_{p,q}) > 0$. From (6.7) we have that $H^{2p-2,p}(D_{p,q}) = H^{2p-2,p}(D_{p,p+1})$. Furthermore we have

Lemma 2 $H^{2p-2,p}(D_{p,p}) = H^{2p-2,p}(D_{p,p+1})$.

Proof:

In order to prove this, we will start from the diagram $D_{p,p+1}$ and we will use the same process as in the proof of Theorem 20. Namely, we obtain the long exact sequences, see (6.1), for every $i = 1, \dots, p-1$:

$$\begin{aligned}\cdots \to H^{2p-3,p-1}(E^i_{p,p+1}) \to H^{2p-2,p}(D^{i-1}_{p,p+1}) \to \\ \to H^{2p-2,p}(D^i_{p,p+1}) \to H^{2p-2,p-1}(E^i_{p,p+1}) \to \cdots \end{aligned} \tag{6.9}$$

For every $i = 1, \dots, p-1$ we can calculate explicitly the number of positive and negative crossings of $E^i_{p,p+1}$, and we can find explicitly the positive diagram to which $E^i_{p,p+1}$ is isotopic.

One can easily see that the number of negative crossings of each $E^i_{p,p+1}$ is equal to $2p-2$ and hence the number of positive crossings is equal to $(p-1)(p+1) - i - (2p-2) = p^2 - 2p + 1 - i$. On the other hand, every $E^i_{p,p+1}$ for $i = 1, \dots, p-2$ is isotopic (by a sequence of R2 and R1 moves as explained in the proof of Lemma 1) to the diagram $D^{i-1}_{p-2,p-1}$, while $E^{p-1}_{p,p+1}$ is isotopic to the diagram $D^{p-3}_{p-2,p-1} \coprod U = D_{p-2,p-2} \coprod U$, where by U we denote the unknot. Hence we have that $\mathcal{H}^l(E^i_{p,p+1}) = 0$ for $l < 0$. Also, since $D^i_{p,q}$ is a positive braid knot with p strands and $(p-1)q - i$ crossings, we have that $\mathcal{H}^{0,(p-3)(p-2)-(i-1)\pm 1}(D^{i-1}_{p-2,p-1}) = \mathbb{Z}$ and all other $\mathcal{H}^{0,j}(D^{i-1}_{p-2,p-1})$ are trivial. Hence, we have that the only nontrivial part of the zeroth homology group of $E^i_{p,p+1}$ is given by $\mathcal{H}^{0,(p-3)(p-2)-(i-1)\pm 1}(E^i_{p,p+1}) = \mathbb{Z}$ for $i = 1, \dots, p-2$, and $\mathcal{H}^{0,(p-3)(p-2)-(p-3)\pm 1 \pm 1}(E^{p-1}_{p,p+1}) = \mathbb{Z}$. Thus, we have that for every $i = 1, \dots, p-1$, $\mathcal{H}^{0,p^2-5p+4-i}(E^i_{p,p+1})$ is trivial.

Finally, since the number of negative crossings of $E^i_{p,p+1}$ is equal to $2p-2$, we have that $H^{2p-3}(E^i_{p,p+1})$ is trivial. Furthermore, since the number of positive crossings of $E^i_{p,p+1}$ is equal to $p^2 - 2p + 1 - i$ we have that

$$H^{2p-2,p-1}(E^i_{p,p+1}) = \mathcal{H}^{0,p^2-5p+4-i}(E^i_{p,p+1})$$

which is trivial for every $i = 1, \dots, p-1$.

Hence from the long exact sequences (6.9) we obtain

$$H^{2p-2,p}(D_{p,p}) = H^{2p-2,p}(D_{p,p+1}),$$

as required. ■

Conjecture 25 *The rank of the homology group $H^{2p-2,p}(D_{p,p})$ (and equivalently of $H^{2p-2,p}(D_{p,p+1})$) is nonzero.*

As we saw, the validity of Conjecture 25 implies that the homological width of the torus knot $T_{p,q}$ is at least p.

Even though we don't (yet) have the proof of Conjecture 25, there is evidence that it is true. First of all, the computer program calculations show that the conjecture is true at least for $p \leq 7$ (the calculations are mainly for knots, i.e. for $D_{p,p+1}$). Furthermore, Lee's variant $H_L^{i,j}$ of Khovanov homology ([34]) for the p-component link $D_{p,p}$ has $2p$ generators in the homological degree $2p-2$. Also, as is well-known, there exist spectral sequences whose E_∞-page is Lee's homology and whose E_2-page is Khovanov homology (see [42], [56]). So $H_L^{i,j} \subset H^{i,j}$ and hence $H^{2p-2}(D_{p,p})$ has at least $2p$ generators. So, we are left with proving that at least one of them has the q-grading equal to p.

6.6 Stable $sl(n)$ homology of torus knots

Define the following normalization of the Poincaré polynomial of the homology of the torus knot:

$$P_{m,n}(t,q) = q^{-(m-1)n} P(T_{m,n})(t,q).$$

Then from the "descending" properties of Theorem 23 we have the following:

Theorem 26 *For every $m \in \mathbb{N}$ there exists a stable homology polynomial P_m^S given by:*

$$P_m^S(t,q) = lim_{n\to\infty} P_{m,n}(t,q).$$

Furthermore, as we have shown, the (normalized) Poincaré polynomial $P_{m,n}(t,q)$ of the torus knot $T_{m,n}$ coincides with the stable polynomial P_m^S, for all powers of t up to $m+n-3$.

Similar results are obtained at the conjectural level in [11] (with a conjectural bound on the powers of q for agreement between the stable homology and the effective homology of any particular torus knot). In [11], reduced homology (see e.g. [28]) is used, but the whole method and all proofs from the previous sections work in the same way for reduced homology.

Also, in [11] the existence of stable $sl(n)$ homology for torus knots is conjectured. However, in the course of proving the stability property in the $sl(2)$ case (Theorem 23, formula (6.6)) we basically used the long exact

sequence in homology. Since the analogous long exact sequence exists for $sl(n)$ homology (it is again the mapping cone - see Remark 13), we can repeat the major part of the process. The long exact sequence in the case of $sl(n)$ homology is:

$$\cdots \to H^{i-1,j+1}(D_1) \to H^{i,j}(D) \to H^{i,j}(D_0) \to H^{i,j+1}(D_1) \to H^{i+1,j}(D) \to \cdots \tag{6.10}$$

where D_i, $i = 0, 1$ is obtained from D after resolving the crossing c into an i-resolution. Note that one of the diagrams D_i is not a planar projection of a knot since it contains one thick edge. In the case that we are interested in (torus knots - positive knots), the diagram D_1 is the one which has one thick edge (for the details and notation see Sections 5.4 and 2.6).

Again, we start from the diagram $D_{p,q}$ of the torus knot $T_{p,q}$, and we resolve the crossing $c_{p-1} = (p-1, 1)$. We denote the diagram obtained by the 0-resolution by $D^1_{p,q}$, and the diagram obtained by the 1-resolution by $\bar{E}^1_{p,q}$. Then we have the following long exact sequence:

$$\cdots \to H_n^{i-1,j+1}(\bar{E}^1_{p,q}) \to H_n^{i,j}(D_{p,q}) \to H_n^{i,j}(D^1_{p,q}) \to H_n^{i,j+1}(\bar{E}^1_{p,q}) \to \cdots$$

We continue the process, by resolving the crossings $c_l = (l, 1)$, $l = p-2, \ldots, 1$ of the diagram $D^{p-1-l}_{p,q}$ and we denote the 0- and 1-resolution obtained, by $D^{p-l}_{p,q}$ and $\bar{E}^{p-l}_{p,q}$, respectively. Then we have:

$$\cdots \to H_n^{i-1,j+1}(\bar{E}^l_{p,q}) \to H_n^{i,j}(D^{l-1}_{p,q}) \to H_n^{i,j}(D^l_{p,q}) \to H_n^{i,j+1}(\bar{E}^l_{p,q}) \to \cdots, \qquad l = 2, \ldots, p-1.$$

Like in the $sl(2)$ case, we shall prove the following

Lemma 3 *The homology group $H^i_n(\bar{E}^l_{p,q})$ is trivial for every $l < p$ and $i < p + q - 3$.*

This lemma, together with the above long exact sequences and the fact that $D^{p-1}_{p,q} = D_{p,q-1}$ gives

$$H_n^{i,j}(D_{p,q-1}) = H_n^{i,j}(D_{p,q}), \quad i < p + q - 3. \tag{6.11}$$

From this formula, we conclude the existence of the limit:

$$\begin{aligned} P^n_k(t,q) &= \lim_{l\to\infty} \sum_{i,j\in\mathbb{Z}} t^i q^j \dim H_n^{i,j}(D_{k,l}) = \\ &= \lim_{l\to\infty} \sum_{i,j\in\mathbb{Z}} t^i q^j q^{(n-1)(k-1)l} \dim \mathcal{H}_n^{i,j}(T_{k,l}) = \\ &= \lim_{l\to\infty} q^{(n-1)(k-1)l} P^n(T_{k,l})(t,q), \end{aligned}$$

for every k, where $P^n(T_{k,l})(t,q)$ is the Poincaré polynomial of the chain complex assigned to $T_{k,l}$ by $sl(n)$-homology. In other words, we obtain

Theorem 27 *There exists stable $sl(n)$ homology for torus knots.*

Thus, we are left with proving Lemma 3. We will use more or less the same approach as in Lemma 1. Let $C_n(\bar{E}^l_{p,q})$ be the chain complex assigned by $sl(n)$-link homology ([30]) to $\bar{E}^l_{p,q}$. Then $H^i_n(\bar{E}^l_{p,q}) = H^i(C_n(\bar{E}^l_{p,q}))$. Note that in the $sl(n)$ case, a complex C_n is assigned to generalized regular diagrams, i.e. to regular diagrams where we also allow trivalent vertices and thick edges.

Since $\bar{E}^l_{p,q}$ has only positive crossings, its homology groups are trivial for negative homological degrees. However, we will show that

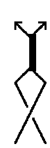
$$C_n(\bar{E}^l_{p,q}) \sim D_n[p+q-3], \tag{6.12}$$

where D_n is a complex such that all its chain groups D^i_n are trivial for $i < 0$ (in fact, we will define D_n as the direct sum of the complexes of the form $C_n(D^i_\Gamma)$, where D^i_Γ's are the generalized regular diagrams whose all crossings are positive). Here, by $\sim$ we denote a quasi-isomorphism, which implies that the two complexes have isomorphic homology groups. Then (6.12) implies that $H^i_n(\bar{E}^l_{p,q}) = H^i(C_n(\bar{E}^l_{p,q}))$ is trivial for $i < p+q-3$.

Like in Lemma 1, we have that the two lower strands (thin edges) of the $\bar{E}_i$ part, will form at least two crossings with each other (corresponding to an R1 move in the proof of Lemma 1) and both of them will have at least $p+q-5$ over- or undercrossings with the same strand (corresponding to an R2 move in the proof of Lemma 1). We will show that in each of these cases we can "shift" up our complex by one homological degree, and thus obtain (6.12).

In order to prove this we will use the following fact ("cancellation principle" for chain complexes): if we quotient the chain complex $\mathcal{C}$ by an (arbitrary) acyclic subcomplex $\mathcal{C}'$ (i.e. a subcomplex with trivial homology), then the quotient complex $\mathcal{C}/\mathcal{C}'$ is quasi-isomorphic to the complex $\mathcal{C}$, and so they have isomorphic homology groups (see e.g. Lemma 3.7 of [5]).

Untwisting an R1 move First, let us work with the analog of the R1 move. Let $\bar{D}$ be the diagram that contains the following diagram as a subdiagram:

D : 

Then (see Section 2.6 and [30]) the chain complex $C_n(\bar{D})$ associated to $\bar{D}$ is the mapping cone of a certain homomorphism $f : C_n(\bar{D}_0) \to C_n(\bar{D}_1)\{-1\}$, where $\bar{D}_0$ and $\bar{D}_1$ are the 0- and 1-resolutions, respectively, of the crossing of D, i.e. they look the same as the diagram $\bar{D}$ except that its subdiagram D is replaced by D_0 and D_1, respectively:

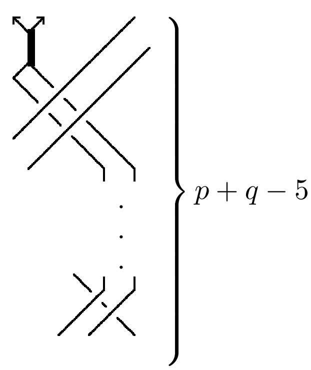
D_0 :  D_1 : 

In other words, the complex associated to $\bar{D}$ is the total complex of the complex given by $C_n(\bar{D}_0) \to C_n(\bar{D}_1)[1]\{-1\}$. Furthermore, we have that (see Section 2.6):

$$C_n(\bar{D}_1) \cong C_n(\bar{D}_0)\{1\} \oplus C_n(\bar{D}_0)\{-1\}.$$

We also have that the projection of f to the first summand is an isomorphism (see [30]), and hence the complex $C_n(\bar{D})$ is quasi-isomorphic to $C_n(\bar{D}_0)[1]\{-2\}$ (by the cancellation principle). So, the last two complexes have isomorphic homology groups. Thus, we can "untwist" the crossing involving two strands that are connected to the same thick edge (the analog of an R1 move in the $sl(2)$ case) by shifting the complex of the diagram obtained up by one in homological degree, as required.

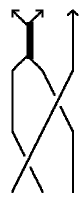
Untwisting an R2 move Hence, after untwisting the two crossings of $\bar{E}^l_{p,q}$ that were resolved in the $sl(2)$ case by the R1 move, we are left with a diagram of the form:



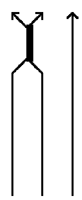
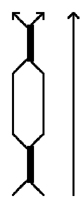
In other words, we have two neighbouring strands that are both connected to the same thick edge, and both go over or both go under $p+q-5$

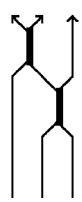
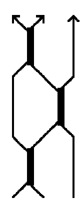
strands. We will show that the complex corresponding to this diagram is quasi-isomorphic to the complex of diagrams whose crossings are all positive, shifted up in homological degree by $p+q-5$.

Let $\bar{D}$ be a positive diagram that contains the following diagram as a subdiagram:

D : 

Denote by D_0, D_1^1, D_1^2 and D_2 the resolutions obtained from D by resolving its two crossings, according to the following pictures:

D_0 :  D_1^2 : 

D_1^1 :  D_2 : 

Denote by $\bar{D}_0$, $\bar{D}_1^1$, $\bar{D}_1^2$ and $\bar{D}_2$ the diagrams obtained from $\bar{D}$, after replacing the subdiagram D by D_0, D_1^1, D_1^2 and D_2, respectively. Then we have that the complex $C_n(\bar{D})$ associated to the diagram $\bar{D}$ is the total complex of the following complex of complexes:

$$\begin{tikzcd} & C_n(\bar{D}_1^1)[1]\{-1\} \arrow[dr] & \\ C_n(\bar{D}_0) \arrow[ur] \arrow[dr] & \bigoplus & C_n(\bar{D}_2)[2]\{-2\} \\ & C_n(\bar{D}_1^2)[1]\{-1\} \arrow[ur] & \end{tikzcd}$$

Like previously, we have that

$$C_n(\bar{D}_1^2) \cong C_n(\bar{D}_0)\{1\} \oplus C_n(\bar{D}_0)\{-1\}, \tag{6.13}$$

and the projection of the map from $C_n(\bar{D}_0)$ onto the first summand of $C_n(\bar{D}_1^2)[1]\{-1\}$ is an isomorphism. Hence, again we can quotient by an acyclic complex and obtain that $C_n(\bar{D})$ is quasi-isomorphic to the total complex of:

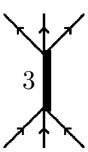
$$\begin{array}{ccc} C_n(\bar{D}_1^1)[1]\{-1\} & & \\ & \searrow & \\ \bigoplus & & C_n(\bar{D}_2)[2]\{-2\} \\ & \nearrow & \\ C_n(\bar{D}_0)[1]\{-2\} & & \end{array} \tag{6.14}$$

Also, we have that

$$C_n(\bar{D}_2) \cong C_n(\bar{D}_0) \oplus C_n(\bar{D}_3), \tag{6.15}$$

("categorification" of the analogous version (2.8) of the fifth axiom from Section 2.5) and that the map from the second summand of (6.13) to the first summand of (6.15) is an isomorphism (see [30]). Here by $\bar{D}_3$ we denoted the diagram that is the same as $\bar{D}$ with the subdiagram D replaced by the following diagram (see Section 2.5):

D_3 :



Hence $C_n(\bar{D})$ is quasi-isomorphic to the total complex of

$$C_n(\bar{D}_1^1)[1]\{-1\} \to C_n(\bar{D}_3)[2]\{-2\} \tag{6.16}$$

On the other hand the complex $C_n(\bar{D}_3)$ is quasi-isomorphic to the total complex of both of the following two complexes:

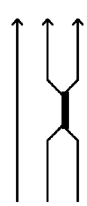
$$C_n(\bar{D}_0)[-1] \to C_n(\bar{D}_2),$$

and

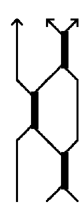
$$C_n(\bar{D}_0')[-1] \to C_n(\bar{D}_2'),$$

where $\bar{D}_0'$ and $\bar{D}_2'$ are the diagrams obtained from $\bar{D}$ after replacing D by following two diagrams, respectively:

D_0' :



D_2' :



Thus, the total complex of (6.14) (and hence $C_n(\bar{D})$) is quasi-isomorphic to the total complex of the following complex:

$$\begin{array}{ccc} C_n(\bar{D}_1^1)[1]\{-1\} & & \\ & \searrow & \\ \bigoplus & & C_n(\bar{D}_2')[2]\{-2\} \\ & \nearrow & \\ C_n(\bar{D}_0')[1]\{-2\} & & \end{array} \tag{6.17}$$

Thus, if we denote by D_n the above complex shifted down in homological degree by 1, then we have that $C_n(\bar{D}) \sim D_n[1]$, and all homology groups of D_n are in nonnegative homological degrees, as required (the crossings in three diagrams appearing in (6.17) are all positive). We can now iterate the argument for each instance when an R2 move would occur in the $sl(2)$ case, since the two lower rightmost strands are both connected to the same thick edge in all three diagrams D_1^1, D_0' and D_2', and hence we can continue the process like with the initial diagram D.

Completely analogously, we obtain the same result for the diagram D of the form:

D : 

Thus, we obtain the required shift in homological degree.

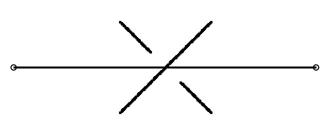
Appendix A

Planar graphs and alternating links

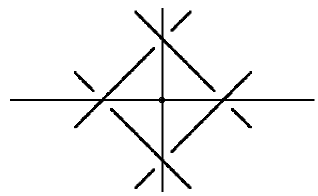
In this appendix we will show that there is a bijection between planar graphs and alternating link diagrams (up to mirror image), and that the Jones polynomial of an alternating link is equal, up to a multiple, to the specialization of the dichromatic polynomial of the corresponding graph. Mainly, we follow [22], section II.8.

In fact, there is a bijective correspondence between planar graphs and link shadows, i.e. the planar projection obtained by replacing each over- and undercrossing with an intersection. Also, every connected link shadow corresponds to two alternating links, one the mirror image of the another, and hence, up to mirror image, every connected planar diagram corresponds to an alternating link.

If we have a planar graph G, then we assign to it the alternating link L^G, in the following way: we replace every edge of the graph G by a crossing according to the following picture



and then we connect the strands near each vertex of the graph, in accordance with the following pattern:



Obviously, in this way we obtain an alternating link.

Conversely, from every alternating link diagram L, we can obtain its shadow by replacing every over- and undercrossing by an intersection (i.e. by a 4-valent vertex). Then we shade the 4-valent graph obtained in this way, in checkerboard fashion (we colour the regions in black and white such that neighbouring regions receive a different colour). Finally, we obtain the corresponding planar graph G^L by assigning a vertex to every black region and every time two black regions touch, or one black region touches itself, at a crossing point of L, we connect the two corresponding vertices, or connect the vertex to itself, with an edge passing through that crossing point of L.

Now, let us see the connection between the Jones polynomial for links and the dichromatic polynomial of graphs. First of all, denote the number of vertices of the graph G by N, and the number of edges of G by m. Fix an ordering of the set of edges, $E(G)$, of the graph G. Then since the crossings of L^G correspond to the edges of G, we obtain an ordering of the crossings of L^G, as well. As we saw in Chapter 3, the dichromatic polynomial, $P_G(q,v)$ of a graph G is given by:

$$P_G(q,v) = \sum_{\epsilon \in \{0,1\}^m} (-1)^{|\epsilon|} q^{|\epsilon|} v^{k(\epsilon)}, \tag{A.1}$$

Here we denote the sum of the entries of ϵ by $|\epsilon|$, and the number of connected components of $[G : s_\epsilon]$ (the spanning subgraph of G whose set of edges is s_ϵ) by $k(\epsilon)$. Recall that we denote by s_ϵ the subset of the set of edges of G such that the i-th edge of G belongs to s_ϵ if and only if $\epsilon_i = 1$. As we saw previously, to every edge e of the graph G we assign a crossing c_e of L^G. In addition, the graph $G - e$ is naturally associated to a 0-resolution of L^G, i.e. the diagram that is obtained by resolving the crossing c_e into a 0-resolution, and the graph G/e is naturally associated to a 1-resolution of L^G, i.e. the diagram that is obtained by resolving the crossing c_e into a 1-resolution. Hence, every $\epsilon \in \{0,1\}^m$ naturally corresponds to a total resolution of L^G, denoted by L^G_ϵ, such that the i-th crossing is resolved into an ϵ_i-resolution.

As we saw in Section 2.3, the Jones polyomial of L^G, $J(L^G)(z)$, is, up to

a multiple, equal to the Kauffman bracket of L^G, $\langle L^G\rangle(z)$, which, according to formula 2.2, is given by:

$$J(L^G)(z) \sim \langle L^G\rangle(z) = \sum_{\epsilon\in\{0,1\}^m} (-1)^{|\epsilon|} z^{|\epsilon|}(z+z^{-1})^{c(\epsilon)}. \tag{A.2}$$

Here by $\sim$ we denote equality up to a multiple, and by $c(\epsilon)$ we denote the number of circles in the total resolution L^G_ϵ. Although $c(\epsilon)$ and $k(\epsilon)$ are not equal in general, there is a simple relation between them. Indeed from the Euler formula for planar graphs, one can easily obtain that:

$$k(\epsilon) = \frac{1}{2}\left(N - |\epsilon| + c(\epsilon)\right). \tag{A.3}$$

For more details see [22], section II.8.

By substituting (A.3) into (A.1) we obtain:

$$\begin{aligned} P_G(q,v) &= \sum_{\epsilon\in\{0,1\}^m} (-1)^{|\epsilon|} q^{|\epsilon|} v^{k(\epsilon)} = \\ &= v^{N/2} \sum_{\epsilon\in\{0,1\}^m} (-1)^{|\epsilon|} (qv^{-1/2})^{|\epsilon|} (v^{1/2})^{c(\epsilon)}. \end{aligned}$$

Now, if we take the specialization given by $v = q^2/(q-1)$ and if we denote $z = (q-1)^{1/2}$, then we have:

$$qv^{-1/2} = (q-1)^{1/2} = z,$$

and

$$v^{1/2} = \frac{q}{(q-1)^{1/2}} = (q-1)^{1/2} + \frac{1}{(q-1)^{1/2}} = z + z^{-1}.$$

Finally, by using (A.2) we have:

$$P_G(q, q^2/(q-1)) \sim \sum_{\epsilon\in\{0,1\}^m} (-1)^{|\epsilon|} z^{|\epsilon|}(z+z^{-1})^{c(\epsilon)} \sim J(L^G)(z),$$

which gives the required relation between this specialization of the dichromatic polynomial of a graph and the Jones polynomial of the corresponding alternating link.

Also, since we know from Chapter 3 (formula (3.1)), that the Tutte polynomial of a graph G is, up to a multiple and a change of variables, equal to the dichromatic polynomial of a graph G, we have that

$$T_G(x,y) \sim P_G(q,v),$$

where $q = 1 - y$ and $v = (x - 1)(y - 1)$, or equivalently $y = 1 - q$ and $x = 1 - v/q$. Hence, the specialization $v = q^2/(q - 1)$ corresponds to the case $x = 1 - q/(q - 1) = 1/(1 - q)$ and $y = 1 - q = 1/x$.

Appendix B

State-sum models for the HOMFLYPT polynomial

In this appendix we will start from the most general state-sum models involving thick edges (of the form (2.10)-(2.11)), and we will show that in this way we obtain two different diagrammatics for the HOMFLYPT polynomial of Sections 2.5 and 4.1. Also, as we shall see, these are essentially the only two different possibilities.

Let L be an oriented link, and let D be a braid presentation of L, i.e. D is the closure of a braid σ, such that D is a regular diagram of L. Denote the number of positive crossings of D by n_+, the number of negative crossings by n_- and the number of strands of the braid σ by $s(D)$. We will define the value of an (intermediate) invariant $\langle D\rangle$ by first giving the value of the bracket on planar trivalent graphs with thick edges, and then we defining its value on (generalized) diagrams via the following recursive relations:

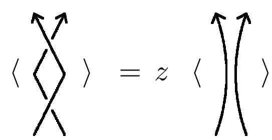
$$\langle D_+\rangle = a\langle D_0\rangle + b\langle D_\Gamma\rangle \tag{B.1}$$
$$\langle D_-\rangle = c\langle D_0\rangle + d\langle D_\Gamma\rangle. \tag{B.2}$$

Here, by D_+, D_-, D_0 and D_Γ we denote the following crossings and resolutions:

D_+ D_- D_0 D_Γ

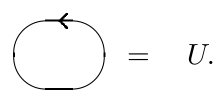
We will not require that the bracket is invariant under the (oriented) Reidemeister moves, but rather that it has the following simple behaviour:

(B.3)



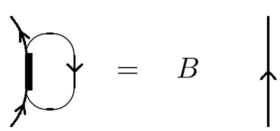
(B.4)

while it is invariant under the R3 move. We don't require any relations for the diagrams related by the Reidemeister 2B move, i.e. the move like in figure (B.4) but with one of the strands oriented downwards, since that situation cannot occur for a diagram which is the closure of a braid diagram. Here and throughout this appendix, by the above relations, we mean relations between two diagrams that look the same except near one crossing where they are as in the above pictures. We denote the value of the free circle by U:



(B.5)

Then from (B.1)-(B.3) we have that the following relation holds:

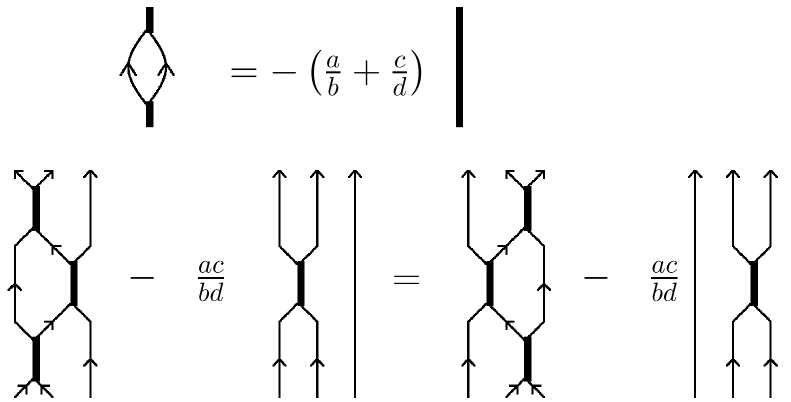


(B.6)

i.e. the values of the bracket on these two diagrams are proportional, and we denote the coefficient of proportionality by B. Hence, we have:

$$
\begin{aligned}
x &= aU + bB, \\
y &= cU + dB.
\end{aligned}
$$

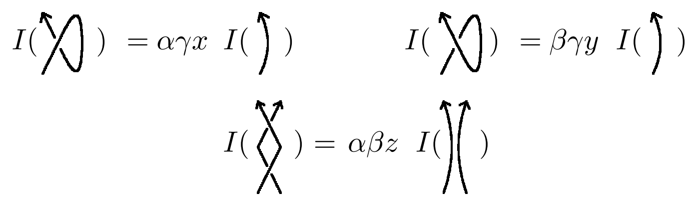
Also, because of the recursive relations (B.1) and (B.2), the bracket will satisfy the relation (B.4) (with $z = ac$), and it will be invariant under the R3 move, if and only if the values of the bracket on trivalent graphs with thick edges satisfy the following axioms:



We will obtain the invariant $I(D)$ as an overall multiple of the bracket:

$$I(D) = \alpha^{n_+}\beta^{n_-}\gamma^{s(D)}\langle D\rangle. \tag{B.7}$$

We determine the coefficients α, β and γ from the requirement that they must compensate the values x, y and z from the behaviour of the bracket under the Reidemeister moves . Thus, from (B.7), (B.3) and (B.4) we obtain:



So, we define α, β and γ such that the above three coefficients are equal to 1, i.e.:

$$\alpha = \left(\frac{y}{xz}\right)^{1/2} = \left(\frac{cU + dB}{ac(aU + bB)}\right)^{1/2}, \tag{B.8}$$

$$\beta = \left(\frac{x}{yz}\right)^{1/2} = \left(\frac{aU + bB}{ac(cU + dB)}\right)^{1/2}, \tag{B.9}$$

$$\gamma = \left(\frac{z}{xy}\right)^{1/2} = \left(\frac{ac}{(aU + bB)(cU + dB)}\right)^{1/2}. \tag{B.10}$$

Note that:

$$(a\alpha)(c\beta) = ac/z = 1. \tag{B.11}$$

Finally, let us see what skein relation the invariant I satisfies. First of all, from the recursive relations (B.1) and (B.2), the bracket $\langle , \rangle$ satisfies the following relation:

$$d\langle D_+\rangle - b\langle D_-\rangle = (ad - bc)\langle D_0\rangle.$$

Thus, from the definition of $I(D)$ (B.7), we obtain:

$$\frac{d}{\alpha}I(D_+) - \frac{b}{\beta}I(D_-) = (ad - bc)I(D_0).$$

After dividing both sides of the above equality by $(abcd)^{1/2}$, denoting $q = (ad/bc)^{1/2}$, and bearing in mind (B.11), we get:

$$(q\beta c)\ I(D_+) - (q\beta c)^{-1}\ I(D_-) = (q - q^{-1})\ I(D_0). \tag{B.12}$$

Thus, we have shown that $I(D)$ satisfies the skein relation of the HOMFLYPT polynomial.

Remark 28 *Note that q and*

$$\beta c = \left(\frac{U + \frac{b}{a}B}{U + \frac{d}{c}B}\right)^{1/2},$$

only depend on the values U, B and the quotients b/a and d/c. Also note that we would obtain the same invariant, if we defined the skein relations for the bracket (B.1) and (B.2) such that a and c are replaced by 1, and b and d by b/a and d/c, respectively, and then defined the invariant $I(D)$ by (B.7) with α replaced by $a\alpha$ and β replaced by $c\beta$.

So, from now on, we will assume that $a = c = 1$, and hence $d = q^2b$. In order to obtain the $sl(n)$ specialization of the HOMFLYPT polynomial, we must have (see (B.12)):

$$q\beta c = q^{\pm n} \tag{B.13}$$

In the case when we have the plus sign, we obtain the following relation:

$$U + bB = q^{2n-2}(U + bq^2B),$$

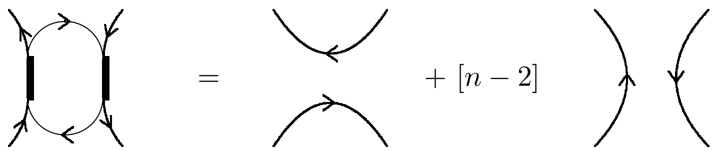
or equivalently:

$$-q^{-1}[n-1]U = [n]bB.$$

Here, by $[n]$ we have denoted the quantum integer n, i.e. $[n] = (q^n - q^{-n})/(q - q^{-1})$. Hence, we have a natural solution:

$$U = [n], \quad B = [n-1], \quad b = -q^{-1}, \quad d = q.$$

In this case, the parameter γ in the definition of the invariant I is equal to 1, and if we add the following requirement



This can be easily seen to imply invariance under the Reidemeister 2B move (the move with diagram as in the picture (B.4) when the orientations of the two strands are opposite). Hence, this model works for arbitrary regular diagrams, and not only for the closures of braid diagrams, and this is the model used for the definition of the HOMFLYPT polynomial in Section 2.5 of Chapter 2.

In the case of the minus sign in (B.13), we have:

$$U + bB = q^{-2n-2}(U + bq^2 B),$$

which gives

$$U(1 + q^2 + q^4 + \ldots + q^{2n}) = -bq^2 B(1 + q^2 + \cdots + q^{2(n-1)}).$$

The last equation has the solution

$$U = 1 + q^2 + \ldots + q^{2(n-1)}, \quad B = 1 + q^2 + \ldots + q^{2n}, \quad b = -q^{-2}, \quad d = -1.$$

In this model, the value of γ is not equal to 1, and hence the invariant $I(D)$ can be defined only by starting from diagrams that are the closures of braid diagrams. Also, following Remark 28, if we define $a = q^2$ and $c = q^{-2}$, and multiply b and d accordingly, we obtain the model used in Section 4.1 of Chapter 4 for the categorification of the n-specializations of the HOMFLYPT polynomial.

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